

## SIMPLE-REGULAR EXPRESSIONS AND LANGUAGES <sup>1 2</sup>

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### ABSTRACT

We define simple-regular expressions and languages. Simple-regular languages provide a necessary condition for a language to be outfix-free. We design algorithms that compute simple-regular languages from finite-state automata. Furthermore, we investigate the complexity blowup from a given finite-state automaton to its simple-regular language automaton and show that there is an exponential blowup. In addition, we present a finite-state automata construction for simple-regular expressions based on state expansion.

*Keywords:* Simple-regular languages, descriptonal complexity, outfix-freeness and state expansion

### 1. Introduction

It is well known that the family of languages specified by finite-state automata (FAs) is the same as the family of languages described by regular expressions [17]. It can be proved by showing that we can construct FAs from regular expressions and that we can construct regular expressions from FAs. Some automata constructions preserve the

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<sup>1</sup>Full version of a submission presented at the 7th Workshop on *Descriptonal Complexity of Formal Systems* (Como, Italy, June 30 – July 2, 2005).

<sup>2</sup>Part of this research was carried out while Han and Trippen were in HKUST.

structural properties of given regular expressions; examples are the Thompson construction [21] and the position construction [11, 18]. Giammarresi et al. [10] examined the structural properties of the Thompson automata and Caron and Ziadi [5] studied the structural properties of the position automata. Giammarresi et al. [9] introduced *Thompson languages* and *simple Thompson languages* based on the structural properties of Thompson automata. One interesting property of simple Thompson languages is that for a given Thompson automaton  $A$ , a string in a simple Thompson language of  $A$  corresponds to a simple path from the start state to the final state of  $A$ .

A code is a set of strings and, thus, is a language. A code can be classified by properties such as *prefix-freeness*, *suffix-freeness*, *infix-freeness* and *outfix-freeness* [2, 16]. The conditions that classify code types define proper subfamilies of given language families. For regular languages, for example, outfix-freeness defines the family of outfix-free regular languages, which is a proper subfamily of regular languages. Since a regular language is defined by an FA, we can classify FAs based on these conditions according to the languages that FAs define. We observe that an FA must have some structural properties to satisfy a certain condition. For example, a deterministic finite-state automaton (DFA)  $A$  should have no out-transitions from all final states if  $A$  defines a prefix-free language, where  $A$  has no sink states [12]. Given a non-deterministic finite-state automaton (NFA)  $A$ , if  $L(A)$  is outfix-free, then there are no cycles in  $A$  since an outfix-free regular language is always finite [16]. In other words, all accepting paths of an outfix-free regular language in  $A$  must be simple. Furthermore, since an outfix-free regular language  $L$  is finite, there is an acyclic deterministic finite-state automaton for  $L$ . Han and Wood [13] designed an algorithm for determining the outfix-freeness of a given acyclic deterministic finite-state automaton based on the structural properties of outfix-free languages.

Note that FAs are directed graphs with labels on edges and simple paths of FAs are already used in the literature [9, 16]. This motivates us to examine a set of strings accepted by simple paths in FAs. In Section 2, we define some basic notions. In Section 3, we introduce *simple-regular expressions* and *languages* and design algorithms for computing simple-regular expressions and languages. The algorithm for computing simple-regular language of a given FA is based on the algorithm [20] for enumerating all simple paths. Then, we investigate the complexity blowup from an FA  $A$  to its simple-regular language FA  $A'$ . We also consider when  $A$  is a Thompson automaton or a position automaton.

## 2. Preliminaries

Let  $\Sigma$  denote a finite alphabet of characters and  $\Sigma^*$  denote the set of all strings over  $\Sigma$ . A language over  $\Sigma$  is any subset of  $\Sigma^*$ . The symbol  $\emptyset$  denotes the empty language and the symbol  $\lambda$  denotes the null string. Given two strings  $x$  and  $y$  in  $\Sigma^*$ ,  $x$  is said to be an *outfix* of  $y$  if there is a string  $w$  such that  $x_1wx_2 = y$ , where  $x = x_1x_2$ . For example,  $abe$  is an outfix of  $abcde$ . Given a set  $X$  of strings,  $X$  is *outfix-free* if no string in  $X$  is an outfix of any other string in  $X$ .

An FA  $A$  is specified by a tuple  $(Q, \Sigma, \delta, s, F)$ , where  $Q$  is a finite set of states,  $\Sigma$  is an input alphabet,  $\delta \subseteq Q \times \Sigma \times Q$  is a transition function,  $s \in Q$  is the start state

and  $F \subseteq Q$  is a set of final states. If  $F$  consists of a single state  $f$ , we use  $f$  instead of  $\{f\}$  for simplicity. Let  $|Q|$  be the number of states in  $Q$  and  $|\delta|$  be the number of transitions in  $\delta$ . Then, the size  $|A|$  of  $A$  is  $|Q| + |\delta|$ . Given a transition  $(p, a, q)$  in  $\delta$ , we say that  $p$  has an *out-transition* and  $q$  has an *in-transition*. Furthermore,  $p$  is a *source state* of  $q$  and  $q$  is a *target state* of  $p$ . We define  $A$  to be *non-returning* if the start state of  $A$  does not have any in-transitions and  $A$  to be *non-exiting* if all final states of  $A$  do not have any out-transitions. We assume that  $A$  has only *useful* states; that is, each state appears on some path from the start state to some final state and, therefore, there is no sink state and  $A$  may not be complete in general. A string  $x$  over  $\Sigma$  is accepted by  $A$  if there is a labeled path from  $s$  to a final state in  $F$  that spells out  $x$ . The language  $L(A)$  of an FA  $A$  is the set of all strings spelled out by paths from  $s$  to a final state in  $F$ .

Given an FA  $A = (Q, \Sigma, \delta, s, F)$  and a state  $q \in Q$ , we define the *right FA*  $A_{\vec{q}}$  to be  $(Q, \Sigma, \delta, q, F)$ ; namely, we make  $q$  to be the start state. Then, the *right language*  $L_{\vec{q}}$  of  $q$  is the set of strings accepted by  $A_{\vec{q}}$ .

For complete background knowledge in automata theory, the reader may refer to textbooks [14, 22].

### 3. Simple-Regular Languages

#### 3.1. Simple-Regular Languages from Regular Expressions and FAs

**Definition 1** Given a regular expression  $E$ , we define the *simple-regular expression*  $\mathcal{S}(E)$  of  $E$  as follows:

1.  $\mathcal{S}(\emptyset) = \emptyset$ .
2.  $\mathcal{S}(\lambda) = \lambda$ .
3.  $\mathcal{S}(a) = a$ , for  $a \in \Sigma$ .
4.  $\mathcal{S}(E + F) = \mathcal{S}(E) + \mathcal{S}(F)$ , where  $E$  and  $F$  are regular expressions.
5.  $\mathcal{S}(E \cdot F) = \mathcal{S}(E) \cdot \mathcal{S}(F)$ .
6.  $\mathcal{S}(E^*) = \lambda + \mathcal{S}(E)$ .

Given a regular expression  $E$ ,  $L(\mathcal{S}(E))$  is the corresponding simple-regular language of  $L(E)$ . A language is often given by an FA. We define a simple-regular language from a FA as follows.

**Definition 2** Given an FA  $A$ , we define the *simple-regular language*  $\mathcal{S}(L(A))$  of  $L(A)$  to be a subset of  $L(A)$  such that each string is accepted by a simple path in  $A$ .

#### 3.2. Simple-Regular Languages from Finite-State Automata

We define a path to be *simple* if it does not have a cycle in an FA  $A$ . Brüggemann-Klein and Wood [3] defined the *orbit* of a state  $q$  and its *gate* states to characterize one-unambiguous regular languages. The orbit  $\mathcal{O}(q)$  of  $q$  in  $A$  is the strongly connected component of  $q$ ; that is, it is the set of states of  $A$  that can be reached from  $q$  and

from which  $q$  can be reached. We say that  $\mathcal{O}(q)$  is *trivial* if it consists of only  $q$  and there are no transitions from  $q$  to itself in  $A$ . A state  $q$  of  $A$  is a gate of its orbit  $\mathcal{O}(q)$  if  $q$  is a final state or  $q$  has an out-transition to a state outside  $\mathcal{O}(q)$ . Note that an orbit is a cycle. If we have  $(p, a, q)$  in  $\delta$ , where  $p \notin \mathcal{O}(q)$ ,  $q \in \mathcal{O}(q)$  and  $a \in \Sigma$ , we call  $p$  an *entry state* of an orbit. Thus, we compute simple paths for each pair of an entry state and a gate state in an orbit in  $A$ .

Given an FA  $A = (Q, \Sigma, \delta, s, F)$ , we transform  $A$  into a new FA  $A'$  such that  $L(A') = L(A)$  and  $A'$  is non-exiting and non-returning as follows:

$$A' = (Q \cup \{s', f'\}, \Sigma, \delta \cup \{(s', \lambda, s)\} \cup \{(f_i, \lambda, f') \mid f_i \in F\}, s', f').$$

Furthermore, if there are more than one transition between two states  $p$  and  $q$  in  $A$ , we combine these transitions as a single transition. For example, if  $(p, a, q)$  and  $(p, b, q)$  in  $\delta$ , then we have  $(p, a + b, q)$  in  $\delta$ .

We now have regular expressions instead of single characters in a transition of  $A$ . We call an FA with regular expressions *expression automaton* (EA). Thus, we have an EA  $A = (Q, \Sigma, \delta, s, f)$  such that there is at most one transition between two states in  $Q$  and  $A$  is non-exiting and non-returning. If a regular expression  $E$  in  $\delta$  is not simple, then we replace  $E$  with its simple-regular expression  $E'$ . If there are self-loops in  $A$ , we remove self-loops since a simple path cannot pass through a self-loop. For a formal definition and more details on EAs, refer to Han and Wood [12]. EAs are a generalization of FAs; they allow regular languages on transitions instead of single characters. If we only allow characters on transitions, then  $A$  is an FA and if we only allow strings, then  $A$  is a generalized automaton [6, 7].

Since the strongly connected components of a directed graph can be computed in linear time [1], we can identify all orbits of  $A$  in  $O(|A|)$  time. Once we identify orbits, then we compute all simple paths for each pair of an entry state and a gate state.

### 3.3. Computing all Simple Paths

We design an algorithm that computes all simple paths between two vertices in a graph based on the algorithm of Rubin [20]. Let  $G = (V, E)$  be a directed graph without multiple edges or self-loops, where  $V$  is a set of vertices and  $E$  is a set of edges. Let  $|V| = n$  be the number of vertices of  $V$  and  $|E| = m$  be the number of edges of  $E$ . A *path* of length  $k$  in  $G$  is a nonempty sequence of vertices  $p = (v_0, v_1, \dots, v_k)$  such that  $(v_i, v_{i+1})$  is an edge of  $G$ . We say a path  $p$  is *simple* if all of its vertices are distinct.

Let  $I_i$  be the  $n$ -bit boolean vector, where 0 is in the position  $i$  and 1 otherwise. Let  $\wedge$  and  $\vee$  be the boolean *and* and *or* operations. Given a path  $p = (v_0, v_1, \dots, v_k)$ , we define the *vertex vector* of  $p$  to be an  $n$ -bit boolean vector such that the bits in positions  $v_0, v_1, \dots, v_k$  are 1 and the others are 0. We define the *edge vector* of  $p$  to be an  $m$ -bit boolean vector such that the bit  $j$  is either 1 if the edge  $e_j$  in  $p$  or 0 otherwise. The *path descriptor* of  $p$  is an ordered pair  $d(p) = (v, e)$ , where  $v$  is the vertex vector and  $e$  is the edge vector of  $p$ .

For example, we can represent the FA in Figure 1 as a graph  $G = (V, E)$  such that  $V = \{1, 2, 3, 4, 5, 6, 7\}$  and  $E = \{e_1 = (1, 2), e_2 = (1, 3), e_3 = (1, 4), e_4 =$

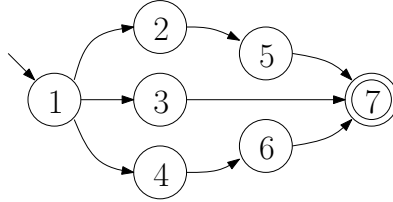


Figure 1: An example of an FA. Note that FAs are essentially graphs with nodes (states) and edges (transitions).

$(2, 5)$ ,  $e_5 = (3, 7)$ ,  $e_6 = (4, 6)$ ,  $e_7 = (5, 7)$ ,  $e_8 = (6, 7)$ }. Note that we assign a unique index number for each state and assign a unique edge index number for each out-transition. We make all out-transition indices of a state to be consecutive in an edge vector of  $G$ ; for example, the first three bits of an edge vector are out-transition indices of state 1 in Figure 1. For a path  $p = 1 \rightarrow 3 \rightarrow 7$ , the path descriptor  $d(p)$  is (1010001, 01001000).

**Lemma 3 (Rubin [20])** *Given a path descriptor  $(v, e)$  in  $G = (V, E)$ , we can compute the sequence of vertices of the path in  $O(m)$  time, where  $m = |E|$ .*

Since  $G$  is an FA  $A$ ,  $m = O(n^2)$  if  $A$  is nondeterministic and we need  $O(n^2)$  time to construct a path from a path descriptor. We propose a more efficient method for computing the sequence of vertices from a given path descriptor in an NFA. Given a path descriptor  $d(p) = (v, e)$  of a simple path  $p$  in  $G$ , where  $v$  is a vertex vector and  $e$  is an edge vector, let us assume that we have computed the vertex sequences from its start vertex  $s$  to an intermediate vertex  $i$  and  $i_o = i_1 i_2 \cdots i_k$  are the out-transition indices of  $i$  in  $e$ . Note that since  $p$  is a simple path only one bit in  $i_o$  must be 1 and the other bits must be 0. We search for the bit with 1.

We notice that bit operations (for example, *shift*, *and*, *or* and *not*) take constant time [19]. We use binary search method to find the bit with 1 from  $i_o = i_1 i_2 \cdots i_k$  using these bit operations. Assume that  $k$  is even. We shift  $i_1 i_2 \cdots i_k$  to the left by  $k/2$  and append a  $k/2$  number of 0s so that we have  $i' = i_{k/2+1} \cdots i_k 000 \cdots 0$ . If  $i' = 0$ , then the bit with 1 is in  $i_1 i_2 \cdots i_{k/2}$ . Otherwise, the bit with 1 is in  $i_{k/2+1} \cdots i_k$ . For example, if  $i$  is 00000100, then  $i' = 01000000$  and we know that the bit with 1 must be in  $i_5 i_6 i_7 i_8 = 0100$ . We repeat this procedure recursively until we find the bit with 1. It takes at most  $\lceil \log k \rceil$  iterations to find the bit with 1 since the procedure is implemented based on binary search method, where  $k$  is the number of out-transitions from  $i$  in  $G$ . The length of a simple path can be at most  $n$  and the number of out-transitions from a vertex can be at most  $n$ .

**Lemma 4** *Given a path descriptor  $(v, e)$  in  $G = (V, E)$ , we can compute the sequence of vertices of the path in  $O(n \log n)$  worst-case time, where  $n = |V|$ .*

Since  $m = O(n^2)$  in NFAs, Lemma 4 is an improvement from  $O(n^2)$  time to  $O(n \log n)$  time compared with Lemma 3.

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**EnumerateAllSimplePath** ( $G = (V, E)$ )

Initialize an  $n \times n$  matrix  $D$ , where  $n = |V|$   
**for**  $(i, j) \in E$   
 $D(i, j) = \{d((i, j))\}$

**for**  $i = 1$  **to**  $n$   
**for**  $j = 1$  **to**  $n$   
**for**  $k = 1$  **to**  $n$   
for each  $(v, e) \in D(j, i)$  and  $(w, f) \in D(i, k)$   
**if**  $v \wedge w \wedge I_i = 0$  **then**  
add  $(v \vee w, e \vee f)$  into  $D(j, k)$

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Figure 2: The algorithm of Rubin [20] that enumerates all simple paths of a given graph  $G$ . Each entry  $D(i, j)$  contains all simple path descriptors from  $i$  to  $j$  in  $G$ .

The algorithm of Rubin [20] is based on matrix multiplication. He showed that we can enumerate all simple paths in a given directed graph  $G$  in  $O(n^3)$  matrix operations<sup>3</sup>, where  $n$  is the number of vertices in  $G$ .

### 3.4. Computing Simple-Regular Languages

Given an EA  $A = (Q, \Sigma, \delta, s, f)$ , we compute the simple path matrix  $D$  for  $A$  using EnumerateAllSimplePath (EASP) in Figure 2. Then, for each pair of an entry state  $i$  and a gate state  $j$  of an orbit, we compute all simple paths from  $D(i, j)$ . For each path descriptor in  $D(i, j)$ , we construct the corresponding simple path and compute the regular expression  $E$  for the simple path. Then, we add a new transition  $(i, E, j)$  into  $\delta$ . After we complete to compute all simple paths for all pairs of an entry state and a gate state of an orbit  $\mathcal{O}(j)$ , we remove all states and transitions in  $\mathcal{O}(j)$  except gate states.

In Figure 3, we construct the simple path matrix using EASP and compute all simple paths from 1 to 2 and from 1 to 3. Note that there is only one nontrivial orbit in the EA  $A$  in Figure 3 (a), where state 1 is an entry state and states 2 and 3 are gate states. Figure 3 (b) illustrates the resulting EA  $A'$  for the simple-regular language of  $L(A)$ .

Note that all regular expressions are concatenations of regular expression in  $A$ ; this implies that there are no Kleene stars since  $A$  has no Kleene stars in transitions. Furthermore, we can transform an EA into a traditional FA using state expansion.

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<sup>3</sup>Note that it is not the total running time to compute all simple paths. There can be an exponential number of simple paths in  $G$ .

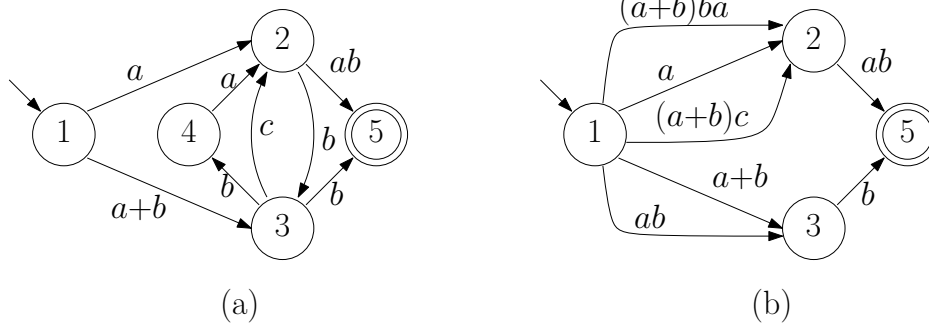


Figure 3: An example of computing the simple-regular language of a given FA using EASP. The set of states  $\{2,3,4\}$  is an orbit  $\mathcal{O}(2)$  and 2,3 are gate states of  $\mathcal{O}(2)$ , where state 1 is an entry state of  $\mathcal{O}(2)$ .

3.5. State Expansion

State elimination was introduced by Brzozowski and McCluskey, Jr. [4] to compute regular expressions from FAs. State elimination maintains the language accepted by a given automaton while removing states; EAs can be used as a data structure for state elimination. *State expansion* is the reverse operation of state elimination.

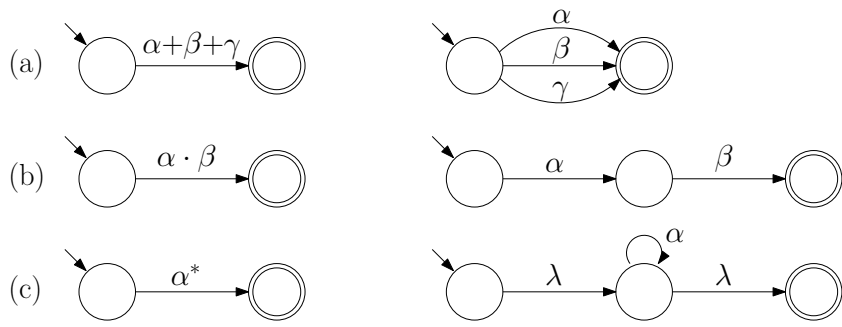


Figure 4: Inductive state expansion procedures; (a)  $E = \alpha + \beta + \gamma$ , (b)  $E = \alpha \cdot \beta$  and (c)  $E = \alpha^*$ , where  $\alpha, \beta$  and  $\gamma$  are regular expressions.

Figure 4 shows the inductive state expansion. We can transform an EA  $A$  into a (traditional) FA by the sequence of state expansions. We expand each regular expression in a transition of  $A$  until the expression is either a single character or the null-string. Ilie and Yu [15] adopted this idea of state expansion and proposed an NFA construction with  $\lambda$ , which is a compact Thompson construction<sup>4</sup>.

Note that the size of the resulting FA by state expansion in Figure 5 has hardly increased from the EA in Figure 3 (b). This leads us to the complexity issues.

<sup>4</sup>Ilie and Yu [15] called the construction the  $\epsilon$ NFA construction, where  $\epsilon$  denotes the null-string  $\lambda$ .

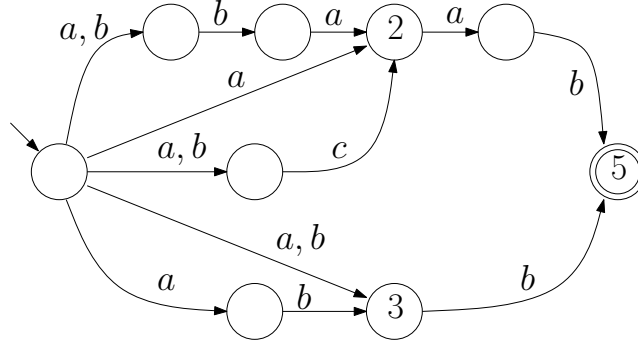


Figure 5: An example of state expansion; the state expansion of the EA in Figure 3 (b).

We investigate the complexity between regular expressions and simple-regular expressions and languages. Let the size  $|E|$  of a regular expression  $E$  be the number of character appearances in  $E$  and  $k$  be the number of Kleene stars in  $E$ .

**Lemma 5**  $|\mathcal{S}(E)| = |E| + k$ .

*Proof.* The proof is straightforward from Definition 1. For each starred subexpression  $F$  in  $E$ , we replace  $F$  with  $\lambda + F$ . For example, if  $E = \alpha \cdot \beta^* \cdot \gamma$ , then  $\mathcal{S}(E) = \alpha \cdot (\beta + \lambda) \cdot \gamma$ . Thus,  $|\mathcal{S}(E)|$  increases by  $k$  if there are  $k$  starred subexpressions in  $E$ .  $\square$

We now consider the size of an FA for  $\mathcal{S}(E)$  based on state expansion. Since  $\mathcal{S}(E)$  does not include any Kleene stars, we use only union and catenation constructions, see Figure 4 (a) and (b).

**Theorem 6** *Given a regular expression  $E$ , let  $A_{\mathcal{S}(E)}$  be the FA of  $\mathcal{S}(E)$  constructed by state expansion as shown in Figure 4. Then,  $|A_{\mathcal{S}(E)}| \leq |E| + 2$ .*

*Proof.* Let  $|E_\Sigma|$  be the number of characters over  $\Sigma$ ,  $|E|$  be the number of catenation operations,  $|E_+|$  be the number of union operations and  $|E_*|$  be the number of Kleene stars in  $E$ , respectively. This implies that  $|E| = |E_\Sigma| + |E| + |E_+| + |E_*|$ . Then,  $|\mathcal{S}(E)| = |E| + |E_*|$  by Lemma 5. Let us compare  $E$  and its simple-regular expression  $\mathcal{S}(E)$ . If a character  $a \in \Sigma$  appears in  $E$ ,  $a$  also appears in  $\mathcal{S}(E)$  and for each Kleene star in  $E$ , there is a corresponding  $\lambda$  in  $\mathcal{S}(E)$ . In other words,  $|\mathcal{S}(E)_\Sigma| = |E_\Sigma| + |E_*|$  and  $|\mathcal{S}(E).| = |E|.|\mathcal{S}(E)$ .

Note that state expansion requires a new state for each catenation operation and a new transition for each character from  $\Sigma$  as shown in constructions (a) and (b) of Figure 4. Therefore,  $|A_{\mathcal{S}(E)}| = |\mathcal{S}(E)_\Sigma| + |\mathcal{S}(E).| + 2 = |E_\Sigma| + |E_*| + |E|.|\mathcal{S}(E) + 2 \leq |E| + 2$ . This concludes the proof.  $\square$

The constant two in Theorem 6 is from the construction. First, we have an EA that has one start state, one final state and one transition between them with a given



regular expression. Theorem 6 shows that state expansion gives an FA with size at most  $|E| + 2$  instead of  $O(|E|)$ , where  $E$  is a simple-regular expression.

3.6. Complexity Issues

We already know that there can be an exponential number of simple paths between two vertices in a graph in the worst-case. On the other hand, an FA that has a polynomial number of states can accept an exponential number of strings. See Figure 6 for an example.

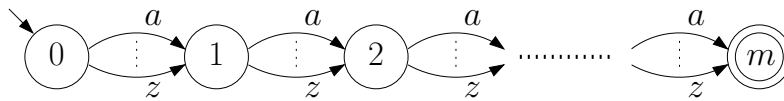


Figure 6: The FA  $A$  accepts all strings of length  $m$  over  $\Sigma = \{a, b, c, \dots, z\}$ ; in other words,  $A$  accepts  $n^m$  strings while the size of  $A$  is  $m + 1 + mn = O(mn)$ .

Thus, one interesting question is that whether or not we need an exponential number of states for representing an FA of simple-regular language of a given FA. Let us consider a DFA  $A = (Q, \Sigma, \delta, s, f)$  such that for any two distinct states  $p$  and  $q$  in  $Q$ ,  $(p, a, q)$  is in  $\delta$  and each transition label in  $\delta$  is unique. See Figure 7 for an example.

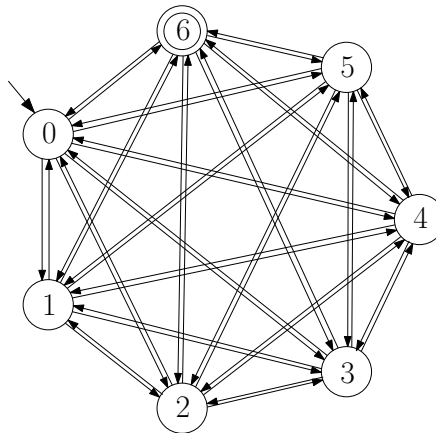


Figure 7: An example of an FA that has an exponential blowup for computing its simple-regular language. The transition label from  $p$  to  $q$  is  $pq$  and, thus, each label is unique. (For example, the transition label from state 1 to state 4 is 14.)

We first construct a DFA  $A' = (Q', \Sigma, \delta', s', f')$  of the simple-regular language of  $L(A)$ . By the definition,  $A' = (Q', \Sigma, \delta', s', f')$  must accept all strings that are spelled out by simple paths in  $A$ . Note that each simple path in  $A$  spells out a unique string since  $A$  has a unique label for each transition. On the other hand, all accepted strings must start with one of out-transition labels of  $s$ . For example,

all strings must start with one of  $\{01, 02, 03, 04, 05, 06\}$  in Figure 7. Let  $m$  be the number of states  $|Q|$  of  $A$ . We add  $|Q| - 1$  out-transitions, which are out-transitions of  $s$  in  $A$ , to the start state  $s'$  and each out-transition label is different from each other. Now let us consider a target state  $q$  of  $s'$ , where  $(s', a, q)$  is in  $\delta'$ . We can find the corresponding state  $q^{\natural}$  in  $A$ ;  $q^{\natural}$  has  $m - 1$  out-transitions and one of them is to  $s$ . Then, with a similar argument, there are only  $|Q| - 2$  out-transitions to  $q$  and all out-transition labels are different. Figure 8 illustrates this procedure for the FA in Figure 7, where  $m = 7$ .

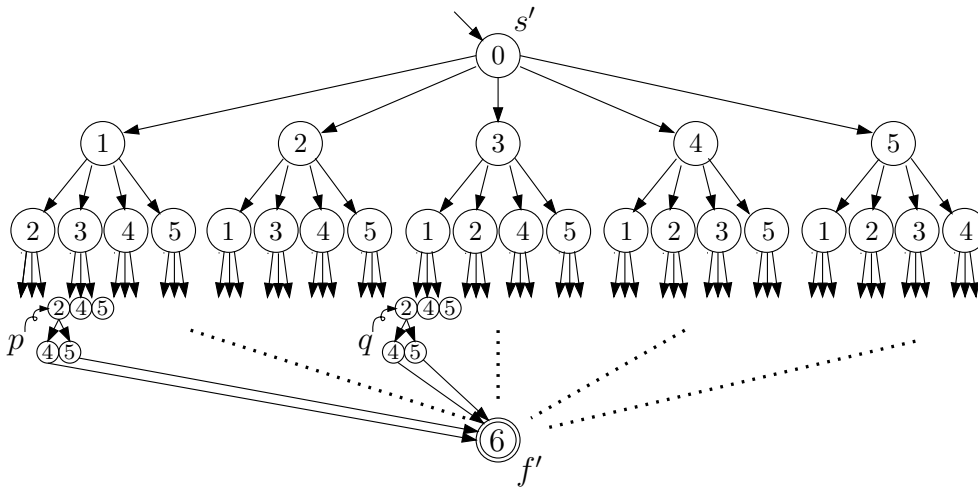


Figure 8: An example of a DFA for a simple-regular language. The indices inside states denote the corresponding states in the FA in Figure 7. We omit all out-transitions from each state to the final state. Note that two states  $p$  and  $q$  are equivalent.

By the construction of  $A'$ , it is easy to verify that  $A'$  is deterministic and  $L(A')$  is the simple-regular language of  $L(A)$ . We define the level of a state  $q$  in  $A'$  to be the minimal number of transitions from  $s$  to  $q$ . Since  $A'$  can accept a string with length  $m - 1$ ,  $L_{\vec{p}}$  of a state  $p$  whose level is  $i$  accepts a string with length  $m - 1 - i$ .

Although  $A'$  is deterministic,  $A'$  is not minimal. We say that a DFA  $A'$  is minimal if and only if  $L_{\vec{p}} \neq L_{\vec{q}}$  for any two states in  $A'$  [22]. If  $L_{\vec{p}} = L_{\vec{q}}$ , then we say  $p$  and  $q$  are *equivalent*; for example, two states  $p$  and  $q$  at the third level in Figure 8 are equivalent.

**Lemma 7** *If two states  $p$  and  $q$  are at different levels in  $A'$ , then  $L_{\vec{p}} \neq L_{\vec{q}}$ .*

*Proof.* Without loss of generality, we assume that the level  $i$  of  $p$  is smaller than the level  $j$  of  $q$ . Note that  $L_{\vec{p}}$  can accept a string with length at most  $m - 1 - j$  and, therefore, a string with length  $m - 1 - i$  cannot be accepted by  $L_{\vec{q}}$  whereas  $L_{\vec{p}}$  can accept a string with length  $m - 1 - i$ . Therefore,  $L_{\vec{p}} \neq L_{\vec{q}}$ .  $\square$

Lemma 7 shows that only states at the same level can be equivalent. Then, we construct the minimal DFA for  $A'$  by identifying equivalent states at each level and merging them. Consider states  $p$  and  $q$  in Figure 8. Let  $\Phi(p)$  be a set of states from  $s'$  to  $p$  in  $A'$  and  $\Phi(p)^{\natural}$  be a set of the corresponding states of  $\Phi(p)$  in  $A$ ; for example,  $\Phi(p)^{\natural} = \{0, 1, 3, 2\}$ . We observe that  $\Phi(p)^{\natural} = \Phi(q)^{\natural}$  and  $p^{\natural} = q^{\natural}$ .

**Lemma 8** *Two states  $p$  and  $q$  in  $A'$  are equivalent if and only if  $\Phi(p)^{\natural} = \Phi(q)^{\natural}$  and  $p^{\natural} = q^{\natural}$ .*

*Proof.*

$\implies$  Assume that  $\Phi(p)^{\natural} \neq \Phi(q)^{\natural}$  or  $p^{\natural} \neq q^{\natural}$ .

1. If  $\Phi(p)^{\natural} \neq \Phi(q)^{\natural}$ , then it implies that the level of  $p$  and the level of  $q$  are different. Then, by Lemma 7,  $L_{\vec{p}} \neq L_{\vec{q}}$  and, therefore,  $p$  and  $q$  are not equivalent – a contradiction.
2. If  $p^{\natural} \neq q^{\natural}$ , then it implies that the label of transition from  $p$  to the final state  $f'$  and the label of transition from  $q$  to  $f'$  are different and, therefore,  $L_{\vec{p}} \neq L_{\vec{q}}$  – a contradiction.

Therefore, if  $p$  and  $q$  are equivalent, then  $\Phi(p)^{\natural} = \Phi(q)^{\natural}$  and  $p^{\natural} = q^{\natural}$ .

$\Leftarrow$  Let  $w$  be a string in  $L_{\vec{p}}$ . Since  $w \in L_{\vec{p}}$ , there is a simple and unique path for  $w$  from  $p^{\natural}$  in  $A$ . Furthermore, this path does not visit any states in  $\Phi(p)^{\natural}$ . Note that  $p^{\natural} = q^{\natural}$  and  $\Phi(p)^{\natural} = \Phi(q)^{\natural}$  and, hence,  $w \in L_{\vec{q}}$ . Therefore, if  $\Phi(p)^{\natural} = \Phi(q)^{\natural}$  and  $p^{\natural} = q^{\natural}$ , then  $L_{\vec{p}} = L_{\vec{q}}$  and  $p$  and  $q$  are equivalent.  $\square$

We investigate the complexity of the minimal DFA  $M(A')$  for  $A'$ . A DFA minimization is based on identifying all equivalent states and merging them into a single state and, thus, a state in  $M(A')$  is a set of equivalent states of  $A'$ . We compute how many equivalent sets of states in  $A'$  to count the number of states in  $M(A')$ .

**Lemma 9** *For  $k \geq 2$ , there are  $\frac{(m-2)(m-3)\dots(m-k-1)}{(k-1)!}$  sets of equivalent states at level  $k$  in  $A'$ .*

*Proof.* For two states  $p$  and  $q$  in  $A'$ , if  $\Phi(p)^{\natural} = \Phi(q)^{\natural}$ , then we denote it by  $p \equiv^{\natural} q$ .

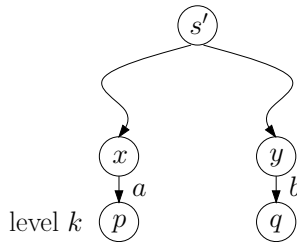


Figure 9: If two states  $p$  and  $q$  are equivalent, then  $x \equiv^{\natural} y$  and  $p^{\natural} = q^{\natural}$  by Lemma 8.

At level  $k$  in  $A'$ , there are  $(m-2)(m-3)\cdots(m-k-1)$  states. Now two states  $p$  and  $q$  among them are equivalent if and only if  $x \equiv^{\natural} y$  and  $p^{\natural} = q^{\natural}$  by Lemma 8, where  $x$  is the source state of  $p$  and  $y$  is the source state of  $q$ . Note that in  $A'$ , each state has a unique source state except  $s'$  and  $f'$ . Let us consider the source state  $x$  of  $p$ . The level of  $x$  is  $k-1$  and it has  $m-k$  out-transitions. There are  $(k-1)!$  states such that for any pair of states  $x$  and  $y$ ,  $x \equiv^{\natural} y$  since there are  $(k-1)!$  permutations for given  $k-1$  characters. Furthermore, if  $(x, a, p)$  is in  $\delta'$ , then there is  $(y, b, q)$  in  $\delta'$  such that  $p^{\natural} = q^{\natural}$ , where  $a$  and  $b$  are not necessary to be same. Note that such  $p$  and  $q$  are equivalent. In other words, there are  $(k-1)!$  equivalent states of  $p$  at level  $k$  including itself. Therefore, there are

$$\frac{(m-2)(m-3)\cdots(m-k-1)}{(k-1)!}$$

sets of equivalent states at level  $k$ . Note that when  $k=1$ , there are  $m-1$  sets of equivalent states instead of  $m-2$  because of the final state  $f'$ .  $\square$

**Theorem 10** *Given a DFA  $A = (Q, \Sigma, \delta, s, f)$ , the size of its minimal DFA of the simple-regular language of  $L(A)$  can be exponential in the size of  $A$  in the worst-case.*

*Proof.* Assume that  $A$  has a similar structure to the FA in Figure 7, where  $|Q| = m$ . Let  $M(A')$  be the minimal DFA of the simple-regular language of  $L(A)$ . Then, there are

$$\frac{(m-2)(m-3)\cdots(m-k-1)}{(k-1)!}$$

states at each level  $k$  in  $M(A')$  for  $1 \leq k \leq m-2$ . (To be precise, the final state is at level 1 but we count the start state and the final state, separately.) Therefore, the number of total states in  $M(A')$  is

$$\begin{aligned} 2 + \sum_{k=1}^{m-2} \frac{(m-2)(m-3)\cdots(m-k-1)}{(k-1)!} \\ = 2 + \sum_{k=1}^{m-2} \binom{m-2}{k-1} (m-k-1) > \sum_{k=1}^{m-2} \binom{m-2}{k-1} = O(2^m). \end{aligned}$$

Hence, there are  $O(2^m)$  states in  $M(A')$ .  $\square$

Theorem 10 shows that there is an exponential blowup for computing the simple-regular language of  $L$  when  $L$  is given by an FA  $A$ . On the other hand, if  $L$  is given by a regular expression  $E$ , then  $|\mathcal{S}(E)| = O(|E|)$  by Definition 1. It leads us to consider the case when  $A$  preserves the structural properties of the corresponding regular expression. For example, the Thompson automata [21] are a proper subfamily of FAs although they represent all regular languages.

**Theorem 11** *Given a Thompson automaton  $A_T$ , we can compute the corresponding Thompson automaton  $A'_T$  such that  $L(A'_T)$  is the simple-regular language of  $L(A_T)$  by deleting back-edges using DFS.*

Note that we can also compute the simple-regular language of a position automaton since both the Thompson automata [21] and the position automata [11, 18] essentially have the same structure for same regular expressions [8].

#### 4. Conclusions

We have introduced simple-regular expressions and languages. Given an outfix-free regular language  $L$  and an FA  $A$  for  $L$ , all strings in  $L$  must be spelled out by simple paths. On the other hand, not all simple-regular languages are outfix-free. For example,  $L(abe + abcde)$  is a simple-regular language but not outfix-free. Thus, simple-regular languages are a proper subfamily of regular languages and a proper superfamily of outfix-free regular languages.

We have designed algorithms that compute an FA  $A'$  from a given FA  $A$  such that  $L(A')$  is the simple-regular language of  $L(A)$ . Since there can be an exponential number of simple paths between two vertices in a graph, we cannot avoid the exponential running time. On the other hand, we know that an FA with a polynomial number of states can accept an exponential number of strings. We have investigated the complexity blowup between  $A$  and  $A'$  and have shown that given an FA  $A$ , the size of its minimal DFA for  $A'$  is exponential in the size of  $A$  in the worst-case, where  $L(A')$  is the simple-regular language of  $L(A)$ . We have also considered the case when  $A$  preserves certain structural properties such as the Thompson automata or the position automata and proved that we can compute the simple-regular languages efficiently without enumerating all simple paths. Since a regular language can be defined by a regular expression, we have presented an efficient FA construction for simple-regular expressions based on state expansion.

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(Received: October 17, 2005; revised: October 17, 2007)