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Outfix-guided insertion ☆

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ABSTRACT

Motivated by work on bio-operations on DNA strings, we consider an outfix-guided insertion operation that can be viewed as a generalization of the overlap assembly operation on strings studied previously. As the main result we construct a finite language L such that the outfix-guided insertion closure of L is non-regular. We consider also the closure properties of regular and (deterministic) context-free languages under the outfix-guided insertion operation and decision problems related to outfix-guided insertion. Deciding whether a language recognized by a deterministic finite automaton is closed under outfix-guided insertion can be done in polynomial time. The complexity of the corresponding question for nondeterministic finite automata remains open.

1. Introduction

Gene insertion and deletion are basic operations occurring in DNA recombination in molecular biology. Recombination creates a new DNA strand by cutting, substituting, inserting, deleting or combining other strands. Possible errors in this process impair the function of genes. Errors in DNA recombination cause mutation that plays a part in normal and abnormal biological processes such as cancer, the immune system, protein synthesis and evolution [1]. Since mutational damage may or may not be easily identifiable, researchers deliberately generate mutations so that the structure and biological activity of genes can be examined in detail. *Site-directed mutagenesis* is one of the most important techniques in laboratory for generating mutations on specific sites of DNA using PCR (polymerase chain reaction) based methods [2,3]. For a site-directed insertion mutagenesis by PCR, the mutagenic primers are typically designed to include the desired change, which could be base addition [4,5]. This enzymatic reaction occurs in the test tube with a DNA strand and predesigned primers in which the DNA strand includes a target region, and a predesigned primer includes a complementary region of the target region. The complementary region of primers leads it to hybridize the target DNA region and generate a desired insertion on a specific site as a mutation. Fig. 1 illustrates the procedure of site-directed insertion mutagenesis by PCR.

In formal language theory, the insertion of a string means adding a substring to a given string and deletion of a string means removing a substring. The insertions occurring in DNA strands are in some sense context-sensitive and Kari and Thierrin [6] modeled such bio-operations using *contextual insertions and deletions* [7,8]. A finite set of insertion–deletion rules, together with a finite set of axioms, can be viewed as a language generating device. Contextual insertion–deletion systems in the study of molecular computing have been used e.g. by Daley et al. [9], Enaganti et al. [10], Krassovitskiy

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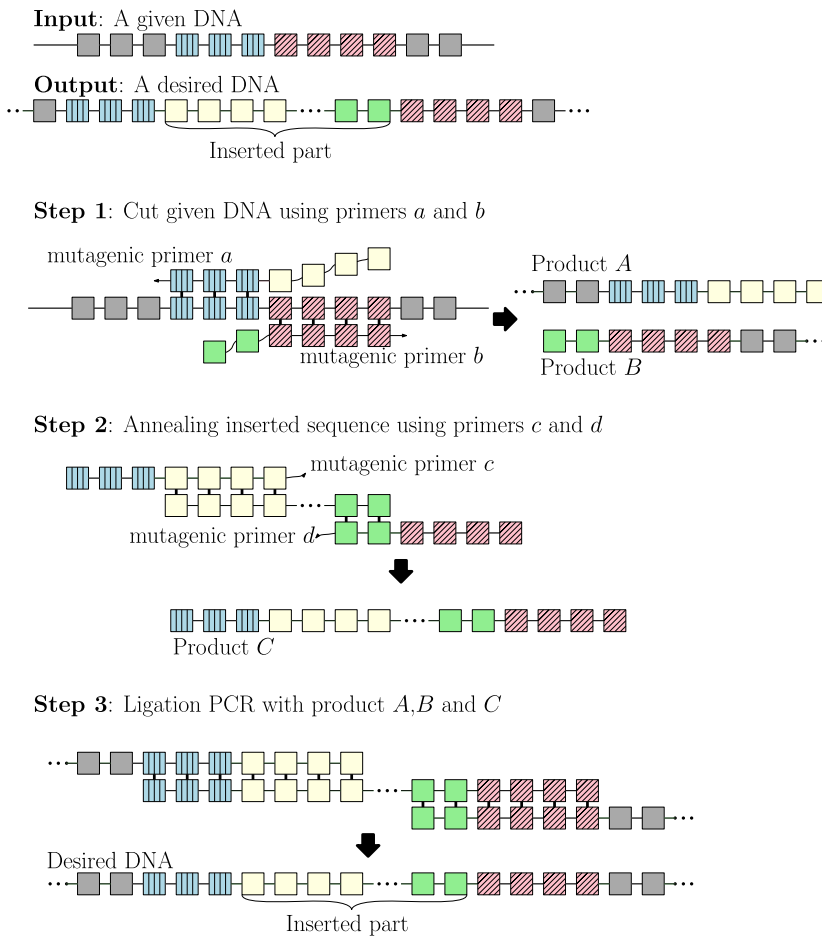


Fig. 1. An example of site-directed insertion mutagenesis by PCR. Given a DNA sequence and four predesigned primers a, b, c and d, two primers a and b lead the DNA sequence to break and extend into two products A and B under enzymatic reaction (Step 1). Two primers c and d complementarily bind to desired insertion region according to the overlapping region and extend into product C (Step 2). Then, the products A, B and C join together to create recombinant DNA that include the desired insertion (Step 3).

et al. [11] and Takahara and Yokomori [12]. Further theoretical studies on the computational power of insertion–deletion systems were done e.g. by Margenstern et al. [13] and Păun et al. [14]. Enaganti et al. [10] have studied related operations to model the action of DNA polymeraze enzymes.

We formalize site-directed insertion mutagenesis by PCR and define a new operation *outfix-guided insertion* that *partially* inserts a string y into a string x when two non-empty substrings of x match with an outfix of y, see Fig. 2(b). We will consider also variants where only a prefix or a suffix of y must match with a non-empty substring of x at the position where the insertion occurs. The outfix-guided insertion is an overlapping variant of the ordinary insertion operation, analogously as the overlap assembly [15–17], cf. Fig. 2(a), is a variant of the ordinary string concatenation operation. An operation equivalent to overlap assembly has been considered under the name chop of languages by Holzer et al. [18]. Holzer and Jacobi [19] have given tight state complexity bounds for a variant of the operation where the overlapping string always has length one. Furthermore, Cărăuşu and Păun [20] have considered another related operation called short concatenation.

This paper investigates the language theoretic closure properties of outfix-guided insertion and iterated outfix-guided insertion. Note that since outfix-guided insertion, similarly as overlap assembly, is not associative, there are more than one way to define the iteration of the operation. We consider a general outfix-guided insertion closure of a language which is defined analogously as the iterated overlap assembly by Enaganti et al. [16]. Iterated (overlap) assembly is defined by Csuhaj-Varju et al. [15] in a different way, which we call right one-sided iteration of an operation.

It is fairly easy to see that regular languages are closed under outfix-guided insertion. Closure of regular languages under iterated outfix-guided insertion turns out to be less obvious. It is well known that regular languages are not closed under the iteration of the ordinary (non-overlapping) insertion operation [21] and it is also fairly easy to establish that iterated prefix-guided (or suffix-guided) insertion does not preserve regularity. However, the known counter-examples, nor their variants, do not work for iterated outfix-guided insertion. Here using a more involved construction we show that there

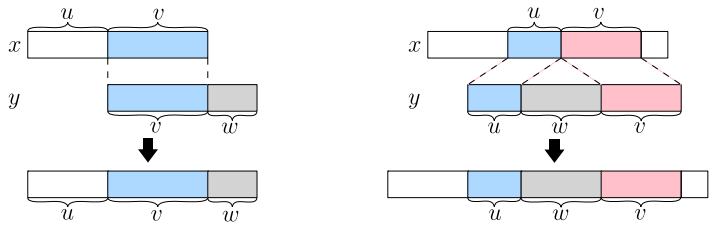


Fig. 2. (a) If suffix v of x overlaps with prefix v of y, then the overlap assembly operation appends suffix w of y to x. (b) If the outfix of y consisting of u and v matches the substring uv of x, then the outfix-guided insertion operation inserts w between u and v in the string x.

exists even a finite language L such that the outfix-guided insertion closure of L is non-regular. On the other hand, we show that the outfix-guided insertion closure of a unary regular language is always regular.

It is well known that context-free languages are closed under ordinary (non-iterated) insertion. We show that context-free languages are not closed under outfix-guided insertion, nor under prefix-guided or suffix-guided insertion. The outfix-guided insertion of a regular language into a context-free language (or vice versa) is always context-free. Also we establish that a similar closure property does not hold for the deterministic context-free and the regular languages. Finally in section 6 we consider decision problems on whether a language is closed under outfix-guided insertion (or og-closed). We give a polynomial time algorithm to decide whether a language recognized by a deterministic finite automaton (DFA) is og-closed. We show that for a given context-free language L the question of deciding whether or not L is og-closed is undecidable.

2. Preliminaries

We assume the reader to be familiar with the basics of formal languages, in particular, with the classes of regular languages and (deterministic) context-free languages [22,23]. Here we briefly recall some definitions and in the next section formally define the main notion of outfix-guided insertion and the corresponding iterated operations.

The symbol Σ stands always for a finite alphabet, Σ^* (respectively, Σ^+) is the set of strings (respectively, non-empty strings) over Σ, $|w|$ is the length of a string $w \in \Sigma^*$, w^R is the reversal of w and ε is the empty string. For $i \in \mathbb{N}$, $\Sigma^{\geq i}$ is the set of strings of length at least i.

If $w = xy$, $x, y \in \Sigma^*$, we say that x is a *prefix* of w and y is a *suffix* of w. If $w = xyz$, $x, y, z \in \Sigma^*$, we say that (x, z) is an *outfix* of w. If additionally $x \neq \varepsilon$ and $z \neq \varepsilon$, (x, z) is a *non-trivial outfix* of w. Sometimes (in particular, when talking about the outfix-guided insertion operation) we refer to an outfix (x, z) simply as a string xz (when it is known from the context what are the components x and z).

Example 2.1. Let $\Sigma = \{a, b, c\}$ and $w = abca$. The non-trivial outfixes of w are (a, a), (ab, a), (a, ca), (a, bca), (ab, ca), and (abc, a). Note that all prefixes and suffixes of a string u are outfixes of u but prefixes and suffixes are not, in general, non-trivial outfixes. A string u represents one or more non-trivial outfixes of u if and only if $|u| \geq 2$.

To conclude this section we fix some basic notation on finite automata.

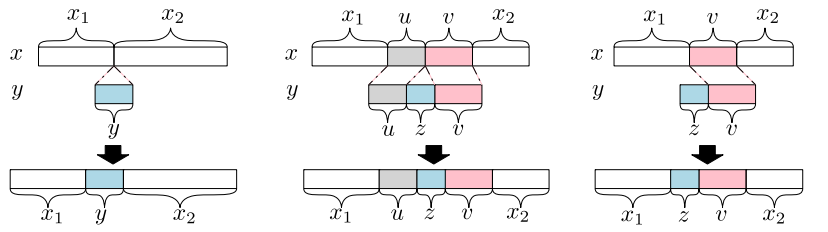
A *nondeterministic finite automaton* (NFA) is a tuple $A = (\Sigma, Q, \delta, q_0, F)$ where Σ is the input alphabet, Q is the finite set of states, $\delta\colon Q \times \Sigma \to 2^Q$ is the transition function, $q_0 \in Q$ is the initial state and $F \subseteq Q$ is the set of final states. In the usual way δ is extended as a function $Q \times \Sigma^* \to 2^Q$ and the *language accepted by* A is $L(A) = \{w \in \Sigma^* \mid \delta(q_0, w) \cap F \neq \emptyset\}$. The automaton A is a *deterministic finite automaton* (DFA) if $|\delta(q, a)| \leq 1$ for all $q \in Q$ and $a \in \Sigma$.

It is well known that the deterministic and nondeterministic finite automata recognize the class of *regular languages*. A (nondeterministic) *pushdown automaton* (PDA) is an extension of a finite automaton that reads the input left-to-right and in addition to the finite state memory has access to a pushdown store [22]. The nondeterministic PDAs define the class of *context-free languages* (CFL). Deterministic PDAs define the class of *deterministic context-free languages* (DCFL) and this is a proper subclass of CFL [22].

3. Definition of (iterated) outfix-guided insertion

We begin by recalling some notions associated with the non-overlapping insertion operation.[1] More details on variants of the insertion operation and iterated insertion can be found in [21].

[1] We use the term "non-overlapping" to make the distinction clear to outfix-guided insertion which will be the main topic of this paper.



(a) non-overlapping insertion (b) outfix-guided insertion (c) suffix-guided insertion

Fig. 3. (a) Non-overlapping insertion of string y into string x. (b) If the outfix of y consisting of u and v matches the substring uv of x, then the outfix-guided insertion operation inserts z between u and v in the string x. (c) If the suffix of y consisting of v matches a substring of x, then the suffix-guided operation inserts the prefix z of y before an occurrence of v in x.

The non-overlapping insertion of a string y into a string x is defined as the set of strings $x \overset{\text{nol}}{\leftarrow} y = \{x_1 y x_2 \mid x = x_1 x_2\}$. The insertion operation is extended in the natural way to languages by setting $L_1 \overset{\text{nol}}{\leftarrow} L_2 = \bigcup_{x \in L_1, y \in L_2} x \overset{\text{nol}}{\leftarrow} y$. Following Kari [21] we define the *left-iterated insertion* of L_2 into L_1 inductively by setting

$$\mathbb{LI}^{(0)}(L_1, L_2) = L_1 \text{ and } \mathbb{LI}^{(i+1)}(L_1, L_2) = \mathbb{LI}^{(i)}(L_1, L_2) \overset{\text{nol}}{\leftarrow} L_2,\ i \geq 0.$$

The *left-iterated insertion closure of* L_2 *into* L_1 is $\mathbb{LI}^*(L_1, L_2) = \bigcup_{i=0}^{\infty} \mathbb{LI}^{(i)}(L_1, L_2)$. It is well known that the iterated non-overlapping insertion operation does not preserve regularity [21,24].

Example 3.1. Let $\Sigma = \{a, b\}$. The left-iterated insertion closure of the string ab into itself is non-regular because $\mathbb{LI}^*(ab, ab) \cap a^*b^* = \{a^i b^i \mid i \geq 0\}$.

Next we define the main notion of this paper which can be viewed as a generalization of the overlap assembly operation [15,16]. An "inside part" of a string y can be outfix-guided inserted into a string x if a non-trivial outfix of y overlaps with a substring of x in a position where the insertion occurs. This differs from contextual insertion (as defined in [6]) in the sense that y must actually contain the outfix that is matched with a substring of x (and additionally [6] specifies a set of contexts where an insertion can occur).

Definition 3.2. The *outfix-guided insertion* of a string y into a string x is defined as

$$x \overset{\text{ogi}}{\leftarrow} y = \{x_1 uzv x_2 \mid x = x_1 uv x_2,\ y = uzv,\ u \neq \varepsilon, v \neq \varepsilon\}.$$

Using the above notations, when $w = x_1 uzv x_2$ is the result of outfix-guided inserting $y = uzv$ into $x = x_1 uv x_2$ we say that the non-empty substrings u and v are the *matched parts*. Note that the matched parts form a non-trivial outfix of the inserted string y. When speaking of matched parts we refer to specific substring occurrences in the string x that are matched with a non-empty prefix and suffix of y, respectively. When string u occurs as a substring x after a prefix of length $i \in \mathbb{N}$, this could be specified as a pair $(u, i+1)$ to indicate that the occurrence begins at position $i+1$.

As variants of outfix-guided insertion we define operations where only a non-empty prefix or a non-empty suffix of the inserted string needs to be matched with a substring in the original string. Naturally it would be possible to define further variants of outfix-guided insertion, e.g., by allowing the matched outfix to be empty.

Definition 3.3. The *prefix-guided insertion* of a string y into a string x is defined as

$$x \overset{\text{pgi}}{\leftarrow} y = \{x_1 y_1 y_2 x_2 \mid x = x_1 y_1 x_2,\ y = y_1 y_2,\ y_1 \neq \varepsilon\}.$$

The *suffix-guided insertion* of a string y into a string x is defined as

$$x \overset{\text{sgi}}{\leftarrow} y = \{x_1 y_1 y_2 x_2 \mid x = x_1 y_2 x_2,\ y = y_1 y_2,\ y_2 \neq \varepsilon\}.$$

The ordinary insertion, outfix-guided insertion and suffix-guided insertion operations, respectively, are illustrated in Fig. 3.

Since we are mainly dealing with outfix-guided insertion, in the following for notational simplicity we write just $\leftarrow$ in place of $\overset{\text{ogi}}{\leftarrow}$. Outfix-guided insertion is extended in the usual way for languages by setting $L_1 \leftarrow L_2 = \bigcup_{w_i \in L_i, i=1,2} w_1 \leftarrow w_2$. The prefix-guided and suffix-guided insertion operations $\overset{\text{pgi}}{\leftarrow}$ and $\overset{\text{sgi}}{\leftarrow}$ are extended for languages in the same way.

It is known that the ordinary insertion operation is not associative and, not surprisingly, neither are the outfix-, prefix- and suffix-guided variants.

Example 3.4. Outfix-guided (respectively, prefix-guided, suffix-guided) insertion operation is not associative.

Let $\Sigma = \{a, b, c, d\}$. Now $abcd \in (acd \leftarrow abc) \leftarrow abcd$ but $abc \leftarrow abcd = \emptyset$.

Similarly we note that $abc \in (ab \overset{\text{pgi}}{\leftarrow} bc) \overset{\text{pgi}}{\leftarrow} abc$ but $bc \overset{\text{pgi}}{\leftarrow} abc = \emptyset$ because no substring of bc is a prefix of abc. By reversing all the strings we get an example that shows that suffix-guided insertion is non-associative.

Since outfix-guided (prefix-guided, suffix-guided, respectively) insertion is non-associative we define the $(i+1)$st iterated operation, analogously as was done with iterated overlap assembly [16], by inserting to a string of the ith iteration another string of the ith iteration.

Definition 3.5. For a language L define inductively

$$\mathbb{OGI}^{(0)}(L) = L, \quad \text{and } \mathbb{OGI}^{(i+1)}(L) = \mathbb{OGI}^{(i)}(L) \leftarrow \mathbb{OGI}^{(i)}(L), \ \ i \geq 0.$$

The *outfix-guided insertion closure* of L is

$$\mathbb{OGI}^{*}(L) = \bigcup_{i=0}^{\infty} \mathbb{OGI}^{(i)}(L).$$

The *prefix-guided insertion closure* of L, $\mathbb{PGI}^{*}(L)$ (respectively, *suffix-guided insertion closure* of L, $\mathbb{SGI}^{*}(L)$) is defined as above by replacing $\leftarrow$ everywhere with $\overset{\text{pgi}}{\leftarrow}$ (respectively, with $\overset{\text{sgi}}{\leftarrow}$).

Recall that the left-iterated non-overlapping insertion [21] discussed above uses two argument languages, and the same is true for the left- and right-iterated outfix-guided insertion introduced below in Definition 3.6. One of the arguments can be viewed as the "target" of the insertions, and the other as the "source" of the inserted strings. The unrestricted insertion closures of Definition 3.5 are defined for one argument language because, roughly speaking, the $i+1$st stage uses the ith stage both as the target and the source of the insertion.

For talking about specific iterated outfix-guided insertions, we use the notation $x \overset{[y]}{\Rightarrow} z$ to indicate that string z is in $x \leftarrow y$, $x, y, z \in \Sigma^{+}$. A sequence of steps

$$x \overset{[y_1]}{\Rightarrow} z_1 \overset{[y_2]}{\Rightarrow} z_2 \overset{[y_3]}{\Rightarrow} \cdots \overset{[y_m]}{\Rightarrow} z_m, \ \ m \geq 1,$$

is called a *derivation* of z_m from x.

When we want to specify the matched substrings, they are indicated by underlining. If $x = x_1 u v x_2$ derives z by inserting uyv (where u and v are the matched prefix and suffix, respectively) this is denoted

$$x_1 \underline{u} v \underline{x_2} \overset{[uyv]}{\Rightarrow} z.$$

Also, sometimes underlining is done only in the inserted string if this makes it clear what must be the matched substrings in the original string.

By a *trivial derivation step* we mean a derivation $x \overset{[x]}{\Rightarrow} x$ where x is obtained from itself by selecting the outfix to consist of the entire string x. Every string of length at least two can be obtained from itself using a trivial derivation step. This means, in particular, that for any language L, $L - (\Sigma \cup \{\varepsilon\}) \subseteq \mathbb{OGI}^{(1)}(L)$. The sets $\mathbb{OGI}^{(i)}(L)$, $i \geq 1$, cannot contain strings of length less than two and, consequently $\mathbb{OGI}^{(i)}(L) \subseteq \mathbb{OGI}^{(i+1)}(L)$, for all $i \geq 1$.

Definition 3.5 iterates the outfix-guided insertion by inserting a string from the ith iteration of the operation into another string in the ith iteration. Since the operation is non-associative we can define iterated insertion in more than one way. The right one-sided iterated insertion of L_2 into L_1 inserts in an outfix-guided way a string of L_2 into L_1 and then iteratively inserts a string obtained in the process into L_1. The left one-sided iterated outfix-guided insertion is defined symmetrically. In fact, when considering iterated ordinary insertion, Kari [21] uses a definition that we call left one-sided iterated insertion (and the operation was defined as $\mathbb{LI}^{*}(L_1, L_2)$ above). Csuhaj-Varju et al. [15] define iterated overlap assembly using right one-sided iteration of the operation.

Definition 3.6. Let L_1 and L_2 be languages. The *right one-sided iterated insertion of L_2 into L_1* is defined inductively by setting $\mathbb{ROGI}^{(0)}(L_1, L_2) = L_2$ and $\mathbb{ROGI}^{(i+1)}(L_1, L_2) = L_1 \leftarrow \mathbb{ROGI}^{(i)}(L_1, L_2)$, $i \geq 0$. The *right one-sided insertion closure* of L_2 into L_1 is $\mathbb{ROGI}^{*}(L_1, L_2) = \bigcup_{i=0}^{\infty} \mathbb{ROGI}^{(i)}(L_1, L_2)$.

The *left one-sided iterated insertion of L_2 into L_1* is defined inductively by setting $\mathbb{LOGI}^{(0)}(L_1, L_2) = L_1$ and $\mathbb{LOGI}^{(i+1)}(L_1, L_2) = \mathbb{LOGI}^{(i)}(L_1, L_2) \leftarrow L_2$, $i \geq 0$. The *left one-sided insertion closure* of L_2 into L_1 is $\mathbb{LOGI}^{*}(L_1, L_2) = \bigcup_{i=0}^{\infty} \mathbb{LOGI}^{(i)}(L_1, L_2)$.

Note that for any language L, $\mathbb{OGI}^{(1)}(L) = \mathbb{LOGI}^{(1)}(L, L) = \mathbb{ROGI}^{(1)}(L, L) = L \leftarrow L$.

The iterated version of unrestricted outfix-guided insertion is considerably more general than the one-sided variants. For any language L, $\mathbb{ROGI}^{*}(L, L)$ and $\mathbb{LOGI}^{*}(L, L)$ are always included in $\mathbb{OGI}^{*}(L)$ and, in general, the inclusions can be strict.

Example 3.7. Let $\Sigma = \{a, b, c\}$ and $L_1 = \{aacc\}$, $L_2 = \{abc\}$. Now $\mathbb{ROGI}^*(L_1, L_2) = a^+bc^+$. For example, by inserting abc into $aacc$ derives $aabcc$:

$$a\underline{ac}c \overset{[\underline{abc}]}{\Rightarrow} aabcc. \tag{1}$$

A right one-sided iterated insertion of L_2 into L_1 could then be continued, for example, as $a\underline{ac}c \overset{[\underline{aabcc}]}{\Rightarrow} aaabcc$. In this way right one-sided derivations can generate all strings of a^+bc^+. Since all inserted strings must contain the symbol b, the first matched part must always belong to a^+ and the second matched part must belong to c^+. This means that $\mathbb{ROGI}^*(L_1, L_2) \subseteq a^+bc^+$.

On the other hand, $\mathbb{LOGI}^*(L_1, L_2) = \{aabcc, aacc\}$. In a left one-sided iterated insertion of L_2 into L_1, the only non-trivial derivation step is (1).

By denoting $L_3 = L_1 \cup L_2$, it can be verified that

$$\mathbb{OGI}^*(L_3) = \mathbb{ROGI}^*(L_3, L_3) = \mathbb{LOGI}^*(L_3, L_3) = a^+bc^+ \cup a^2a^*c^2c^*.$$

The next example illustrates that unrestricted outfix-guided insertion closure of a language L' can be larger than $\mathbb{LOGI}^*(L', L')$. The language L used in the proof of Theorem 4.6 in the next section gives an example where the unrestricted insertion closure is larger than $\mathbb{ROGI}^*(L, L)$ (as explained before Proposition 4.13).

Example 3.8. Let $\Sigma = \{a, b, c, d, e, f\}$ and $L' = \{abce, bcde, acdef\}$. We note that $a\underline{bce} \overset{[\underline{bcde}]}{\Rightarrow} abcde$. Furthermore, it is easy to verify that by outfix-guided inserting strings of L' into $L' \cup \{abcde\}$ one cannot produce more strings and, thus, $\mathbb{LOGI}^*(L', L') = L' \cup \{abcde\}$. On the other hand, we have

$$\underline{acde}f \overset{[\underline{abcde}]}{\Rightarrow} abcdef \in \mathbb{OGI}^{(2)}(L').$$

4. Outfix-guided insertion and regular languages

As can be expected, the family of regular languages is closed under the outfix-guided (prefix-guided, suffix-guided, respectively) insertion operation. On the other hand, the answer to the question whether regular languages are closed under iterated outfix-guided insertion seems less clear. From Kari [21] we recall that it is easy to construct examples that establish the non-closure of regular languages under iterated non-overlapping insertion. Using variants of such examples we see that the prefix-guided (or suffix-guided) insertion closure of a singleton language may be non-regular.

On the other hand, analogous straightforward counter-examples do not work for the unrestricted outfix-guided insertion closure. Using a more involved construction we establish that the outfix-guided insertion closure of a finite language need not be regular. The non-closure of regular languages under right one-sided insertion closure is established by a more straightforward construction (Proposition 4.13).

We begin by showing that regular languages are closed under non-iterated outfix-guided insertion. The proof is not surprising but we give an explicit construction because, essentially, the same construction will be used to show in Theorem 5.3 that the outfix-guided insertion of a regular (respectively, context-free) language into a context-free (respectively, regular) language is always context-free, and for the polynomial time algorithm to decide whether the language recognized by a DFA is closed under outfix-guided insertion in section 6.

Lemma 4.1. *If L_1 and L_2 are regular, then so is $L_1 \leftarrow L_2$.*

Proof. Let L_1 be recognized by an NFA $A = (\Sigma, Q, \delta, q_0, F_A)$ and L_2 be recognized by an NFA $B = (\Sigma, P, \gamma, p_0, F_B)$. Denote $\overline{Q} = \{\overline{q} \mid q \in Q\}$ and $\overline{P} = \{\overline{p} \mid p \in P\}$. Here $Q \cup P$ is disjoint with $\overline{Q} \cup \overline{P}$ and ♣, ♡ are new symbols not occurring in any of the sets.

For the language $L_1 \leftarrow L_2$ we construct an NFA $C = (\Sigma, R, \omega, r_0, F_C)$ where

$$R = Q \times (P \cup \overline{P} \cup \{\clubsuit, \heartsuit\}) \cup \overline{Q} \times P,$$

$$F_C = \{(q, \overline{p}) \mid q \in F_A, p \in F_B\} \cup \{(q, \heartsuit) \mid q \in F_A\},$$

$r_0 = (q_0, \clubsuit)$, and for defining the transitions of ω let b be an arbitrary symbol of Σ. We set

(i) for $q \in Q$: $\omega((q, \clubsuit), b) = \{(q', \clubsuit) \mid q' \in \delta(q, b)\} \cup \{(q', p') \mid q' \in \delta(q, b), p' \in \gamma(p_0, b)\}$,
(ii) for $q \in Q$, $p \in P$: $\omega((q, p), b) = \{(q', p') \mid q' \in \delta(q, b), p' \in \gamma(p, b)\} \cup \{(\overline{q}, p') \mid p' \in \gamma(p, b)\} \cup \{(q', \overline{p'}) \mid q' \in \delta(q, b), p' \in \gamma(p, b)\}$,
(iii) for $q \in Q$, $p \in P$: $\omega((\overline{q}, p), b) = \{\overline{q}, p') \mid p' \in \gamma(p, b)\} \cup \{(q', \overline{p'}) \mid q' \in \delta(q, b), p' \in \gamma(p, b)\}$,

(iv) for $q \in Q$, $p \in P$: $\omega((q, \overline{p}), b) = \{(q', \overline{p'}) \mid q' \in \delta(q, b), p' \in \gamma(p, b)\} \cup Z_p$, where

$$Z_p = \begin{cases} \{(q', \heartsuit) \mid q' \in \delta(q, b)\} & \text{if } p \in F_B, \\ \emptyset & \text{if } p \notin F_B, \end{cases}$$

(v) for $q \in Q$: $\omega((q, \heartsuit), b) = \{(q', \heartsuit) \mid q' \in \delta(q, b)\}$.

All transitions not listed above are undefined.

We begin by verifying that $L(A) \leftarrow L(B) \subseteq L(C)$. Consider a string $w = x_1uzvx_2$ where $x_1uvx_2 \in L(A)$ and $uzv \in L(B)$, $u, v \neq \varepsilon$. Roughly speaking, C uses the states of $Q \times \{\clubsuit\}$ to process the prefix x_1, the states of $Q \times P$ to process the following substring u, the states of $\overline{Q} \times P$ to process the substring z, the states of $Q \times \overline{P}$ to process the substring v, and the states of $Q \times \{\heartsuit\}$ to process the suffix x_2. Note that according to rules (ii), on states of $Q \times P$ the NFA simulates A in the first component and B in the second component of the states. According to rules (iii), on states of $\overline{Q} \times P$, the NFA C simulates only B in the second component, and according to rules (iv), on states of $Q \times \overline{P}$ the NFA C simulates again both A and B (in the first and second component of the state of C, respectively).

In more detail, consider an accepting computation $\mathrm{comp}_A(x_1uvx_2)$ of A on x_1uvx_2 that reaches state q_{x_1} (respectively, q_u, q_v, q_{x_2}) after reading the prefix x_1 (respectively, x_1u, x_1uv, x_1uvx_2). An accepting computation of C first reads x_1 using rules (i) and simulating the computation $\mathrm{comp}_A(x_1uvx_2)$ of A on the prefix x_1, thus ending in state $(q_{x_1}, \clubsuit)$.

When reading the first symbol b_1 of u, again using a rule (i) the computation of C goes to a state (q', p') where $q' \in \delta(q_{x_1}, b_1)$ and the second component begins to simulate an accepting computation of B on uzv in a state $p' \in \gamma(p_0, b_1)$. The computation nondeterministically guesses when it sees the first symbol of z, and using rules (ii) enters a state $(\overline{q_u}, p')$ where p' is the state of B in an accepting computation on uzv after reading the first symbol of z. If $z = \varepsilon$, the computation guesses when it sees the first symbol b_2 of v and using the "third option" in the rules (ii), C goes to a state (q', p') where $q' \in \delta(q_u, b_2)$ and p' is a state that can be reached by B after reading ub_2.

The computation processes the substring z in a state of $\{\overline{q_u}\} \times P$ using rules (iii) and simulating the computation of B in the second component. When the computation guesses that it sees the first symbol b_2 of v, using the "second part" of the rules (iii) the NFA C goes to a state (q', p') where $q' \in \delta(q_u, b_2)$ and p' is a state that can be reached by B after reading the prefix uzb_2. Then, according to rules (iv), C simulates the computation $\mathrm{comp}_A(x_1uvx_2)$ in the first component of the state and B in the second component of the state. Always when the second component is an element of F_B, according to rules (iv), the computation may enter a state of the form $(q', \heartsuit)$ that indicates that it has finished reading the substring uzv.

The remaining computation, using rules (v), simulates the computation $\mathrm{comp}_A(x_1uvx_2)$ of A on the first component of the states. The choice of the final states then guarantees that C accepts in the state $(q_{x_2}, \heartsuit)$. If $x_2 = \varepsilon$, then the computation of C ends in an accepting state (q_v, p_f) where $p_f \in F_B$ is the state at the end of the simulated computation of B on uzv.

For the converse inclusion we note that the definition of the transitions of ω guarantees that any computation of C ending in an accepting state, must have five parts P_1, P_2, P_3, P_4 and P_5, where P_1 uses states of $Q \times \{\clubsuit\}$, P_2 uses states of $Q \times P$, P_3 uses states of $\overline{Q} \times P$, P_4 uses states of $Q \times \overline{P}$ and P_5 uses states of $Q \times \{\heartsuit\}$. The part P_3 may be empty if, according to rules (ii), the computation jumps directly from a state of P_2 to a state of P_4 and the part P_5 may be empty if the computation P_4 ends in an accepting state of the form $(q, \overline{p})$ ($q \in F_A$, $p \in F_B$).

Since states of P_1 and P_5 simulate only a computation of A, states of P_3 simulate only a computation of B and states of P_2 and P_4 simulate both a computation of A and a computation of B, it is easy to verify that C can have accepting computations only on strings of $L(A) \leftarrow L(B)$. □

The result of Lemma 4.1 extends easily using induction:

Proposition 4.2. *Suppose L_1 and L_2 are regular languages. Then, for all $i \geq 0$, $\mathbb{OGI}^{(i)}(L_1)$, $\mathbb{ROGI}^{(i)}(L_1, L_2)$ and $\mathbb{LOGI}^{(i)}(L_1, L_2)$ are regular.*

A simplified variant of the proof of Lemma 4.1 allows us to show that the prefix-guided (or suffix-guided) insertion of a regular language into a regular language is regular. We leave the proof as an exercise.

Proposition 4.3. *If L_1 and L_2 are regular languages, then so are $L_1 \overset{\text{pgi}}{\leftarrow} L_2$ and $L_1 \overset{\text{sgi}}{\leftarrow} L_2$.*

Iterated prefix-guided or suffix-guided insertion does not preserve regularity.

Proposition 4.4. *There exist singleton languages L_1 and L_2 such that $\mathbb{PGI}^*(L_1)$ and $\mathbb{SGI}^*(L_2)$ are non-regular.*

Proof. Choose $L_1 = aab$. We claim that

$$\mathbb{PGI}^*(aab) \cap a^*b^* = \{a^i b^j \mid 2 \leq i \leq j + 1\}. \tag{2}$$

To establish the inclusion from right to left we note that, for all $i \geq 2$, $a^{i+1}b^{i+2} \in a^ib^{i+1} \overset{\text{pgi}}{\leftarrow} aab$. Furthermore, into strings of the form a^ib^j $(i \geq 2)$ one can always add exactly one b by inserting aab where prefix aa is matched with the last a's of a^ib^j.

Second we verify the inclusion from left to right in (2). Denote $Z = \{a^ib^j \mid 2 \leq i \leq j+1\}$. We note that derivations of strings in $\mathbb{PGI}^*(aab)$ belonging to a^*b^* can use only strings in a^*b^* because if $w_1 \notin a^*b^*$ or $w_2 \notin a^*b^*$ then any string in $w_1 \overset{\text{pgi}}{\leftarrow} w_2$ has an occurrence of b preceding an occurence of a. It is clear that for $u_1, u_2 \in Z$, all strings in $(u_1 \overset{\text{pgi}}{\leftarrow} u_2) \cap a^*b^*$ must be in Z. Note that the matched prefix of u_2 must contain at least one a and thus the insertion adds to u_1 at least as many b's as a's.

Since the language Z is non-regular, (2) implies that $\mathbb{PGI}^*(aab)$ is non-regular.

A completely symmetric argument establishes that $\mathbb{SGI}^*(abb)$ is non-regular. □

It seems difficult to extend the proof of Lemma 4.1 for outfix-guided insertion closure because on strings with iterated insertions, the computations on corresponding prefix–suffix pairs can, in general, depend on each other and when processing a part inserted in between, an NFA would need to keep track of such pairs, as opposed to simply keep track of a set of states. On the other hand, constructions as in the proof of Proposition 4.4 rely on the property that the matched substrings are all either prefixes or all suffixes of the inserted strings and this type of straightforward constructions do not yield a regular language whose outfix-guided insertion closure is non-regular.

Next we show that regular languages, indeed, are not closed under iterated outfix-guided insertion. For the construction we use the following technical lemma.

Lemma 4.5. *Let* $\Sigma = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ *and define*

$$L_1 = \{a_3a_1a_2b_1,\ a_2b_2b_1b_3,\ a_1a_2a_3b_2,\ a_3b_3b_2b_1,\ a_2a_3a_1b_3,\ a_1b_1b_3b_2\}.$$

Then $L_1 \leftarrow L_1 = L_1$.

Proof. The inclusion from right to left follows from the observation that any string w of length at least two is a non-trivial outfix of itself and, consequently w can be inserted into itself to produce w as a result.

For the converse inclusion we verify that for all $x, y \in L_1$, if $y \neq x$, then y cannot be outfix-guided inserted into x and x can be outfix-guided inserted into itself only in the trivial way of using as matching parts a non-empty prefix x_1 and a non-empty suffix x_2 such that $x_1x_2 = x$. The second claim is obvious because each string of L_1 consists of 4 different symbols.

Consider now $x, y \in L_1$, $x \neq y$. For the sake of contradiction suppose that $w \in x \leftarrow y$ and that w is obtained by matching substrings u and v of x with a prefix and a suffix of y, respectively. The substrings u and v cannot both consist of symbols a_i, $1 \leq i \leq 3$, because if x and y both contain more than one symbol a_i, they must end with symbols b_{j_1} and b_{j_2}, $j_1 \neq j_2$. Using a symmetric argument we observe that u and v cannot both consist of symbols b_i, $1 \leq i \leq 3$.

The remaining possibility is that the first matched part u consists of (one or more) symbols a_i and the second matched part v consists of (one or more) symbol b_j. For simplicity in the following discussion we assume that v begins with b_1. The definition of L_1 is symmetric, and an analogous argument works when the first symbol of v is b_2 or b_3.

Now uv must be a substring of x. This means that the last symbol of u can be a_2 if $x = a_3a_1a_2b_1$ or a_1 if $x = a_1b_1b_3b_2$. Besides the string $a_3a_1a_2b_1$ the only other string of L_1 where a_2 occurs before b_1 is $a_2b_2b_1b_3$. The string $a_2b_2b_1b_3$ cannot be inserted into $x = a_3a_1a_2b_1$ because the last symbol b_3 would be "outside" of x.

As the remaining case consider then the possibility $x = a_1b_1b_3b_2$. The only string of $L_1 - \{x\}$ where a_1 occurs before b_1 is $a_3a_1a_2b_1$ and again this cannot be inserted into x because the first symbol a_3 would be outside of x. □

Theorem 4.6. *There exists a finite language* L *such that* $\mathbb{OGI}^*(L)$ *is non-regular.*

Proof. Let $\Sigma = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and define $L \subseteq (\Sigma \cup \{\$\})^*$ as

$$L = \{\$a_3a_1b_1b_3\$,\ a_3a_1a_2b_1,\ a_2b_2b_1b_3,\ a_1a_2a_3b_2,\ a_3b_3b_2b_1,\ a_2a_3a_1b_3,\ a_1b_1b_3b_2\}.$$

Note that $L - \{\$a_3a_1b_1b_3\$\}$ is equal to the language L_1 from Lemma 4.5. Our construction is based on an idea that the only way to produce new strings in $\mathbb{OGI}^*(L)$ is to insert into strings obtained from $\$a_3a_1b_1b_3\$$ cyclically copies of the strings of L_1. For ease of discussion we introduce names for the strings of L_1:

$$y_1 = a_3a_1a_2b_1,\ \ y_2 = a_2b_2b_1b_3,\ \ y_3 = a_1a_2a_3b_2,\ \ y_4 = a_3b_3b_2b_1,$$
$$y_5 = a_2a_3a_1b_3,\ \ y_6 = a_1b_1b_3b_2.$$

For specifying the language $\mathbb{OGI}^*(L)$ we define the finite set

$$S_{\text{middle}} = \{a_1b_1,\ a_1a_2b_1, a_1a_2b_2b_1,\ a_1a_2a_3b_2b_1,\ a_1a_2a_3b_3b_2b_1,\ a_1a_2a_3a_1b_3b_2b_1\}.$$

We claim that

$$\mathbb{OGI}^*(L) = \{\$a_3(a_1a_2a_3)^i z(b_3b_2b_1)^i b_3\$ \mid i \geq 0,\ z \in S_{\text{middle}}\}. \tag{3}$$

To establish the inclusion from right to left, we note that

$$\$a_3a_1b_1b_3\$ \overset{[y_1]}{\Rightarrow} \$a_3a_1a_2b_1b_3\$ \overset{[y_2]}{\Rightarrow} \$a_3a_1a_2b_2b_1b_3\$ \overset{[y_3]}{\Rightarrow} \$a_3a_1a_2a_3b_2b_1b_3\$ \overset{[y_4]}{\Rightarrow}$$
$$\$a_3a_1a_2a_3b_3b_2b_1b_3\$ \overset{[y_5]}{\Rightarrow} \$a_3a_1a_2a_3a_1b_3b_2b_1b_3\$ \overset{[y_6]}{\Rightarrow} \$a_3a_1a_2a_3a_1b_1b_3b_2b_1b_3\$ = w_1.$$

The first five insertions generate the strings $\$a_3zb_3\$$, $z \in S_{\text{middle}}$, and the last string w_1 again has "middle part" $a_3a_1b_1b_3$. By cyclically outfix-guided inserting the strings $y_1, \ldots, y_6$ into w_1 we get all strings $\$a_3(a_1a_2a_3)z(b_3b_2b_1)b_3\$$, $z \in S_{\text{middle}}$, and the string $\$a_3(a_1a_2a_3)^2a_1b_1(b_3b_2b_1)^2b_3\$$. By simple induction it follows that $\mathbb{OGI}^*(L)$ contains the right side of (3).

To establish the converse inclusion, we verify that all strings obtained by iterated outfix-guided insertion from strings of L must be obtained as above, that is, all non-trivial derivations producing new strings must be as above.

Since $\$a_3a_1b_1b_3\$$ is the only string in L containing symbols \$ and they occur as the first and the last symbol, it is clear that all strings in $\mathbb{OGI}^*(L)$ containing symbols \$ must be in $\$\Sigma^*\$$. A string of $\$\Sigma^*\$$ cannot be outfix-guided inserted to any string not containing symbols \$ and a string of $\$\Sigma^*\$$ can be outfix-guided inserted into another string of $\$\Sigma^*\$$ only using a trivial derivation step.

By Lemma 4.5 we know that strings of L_1 cannot be outfix-guided inserted into other strings of L_1.

We have verified that the set

$$L_{\text{gen}} = \{\$a_3(a_1a_2a_3)^i z(b_3b_2b_1)^i b_3\$ \mid i \geq 0,\ z \in S_{\text{middle}}\}$$

is included in $\mathbb{OGI}^*(L)$ and strings of L_{gen} can be inserted into strings of L_{gen} only in a trivial way. To complete the proof it remains to verify that inserting strings of L_1 into L_{gen} does not produce additional strings, that is, $L_{\text{gen}} \leftarrow L_1 \subseteq L_{\text{gen}}$.

It is impossible to insert $a_3a_1a_2b_1$ into a string of L_{gen} using an outfix obtained from a_3 and a_2b_1 because in strings of L_{gen} the symbols a_3 and a_2 do not occur consecutively. The same applies to the other five strings $y_2, \ldots, y_6 \in L_1$: we cannot insert y_j into L_{gen} using an outfix where the prefix ends and the suffix begins with a symbol of $\{a_1, a_2, a_3\}$ (respectively, with a symbol of $\{b_1, b_2, b_3\}$).

The other non-trivial possibilities are that we insert $a_3a_1a_2b_1$ into a string of L_{gen} using an outfix uv where u is either a_3 or a_3a_1 and $v = b_1$. The choice $u = a_3$, $v = b_1$ is not possible because a_3b_1 is not a substring of a string in L_{gen}. The insertion using outfix $a_3a_1b_1$ can be done only to a string of the from $\$a_3(a_1a_2a_3)^i a_1b_1(b_3b_2b_1)^i b_3\$$, $i \geq 0$ and it produces $\$a_3(a_1a_2a_3)^i a_1a_2b_1(b_3b_2b_1)^i b_3\$ \in L_{\text{gen}}$. Using symmetry of the definition of L_1, the argument for the other five strings of L_1 is completely analogous.

This establishes (3) and the non-regularity of $\mathbb{OGI}^*(L)$. □

We conjecture that the iterated outfix-guided insertion closure of a regular language need not be even context-free. However, a construction of such a language would seem to be considerably more complicated than the construction used in the proof of Theorem 4.6.

Open problem 4.7. *Find a regular (or a finite) language L such that $\mathbb{OGI}^*(L)$ is not context-free.*

Contrasting the result of Theorem 4.6 we show that unary regular languages are closed under iterated outfix-guided insertion. The construction is based on a technical lemma which shows that, for unary languages, outfix-guided insertion closure can be represented as a variant of the iterated overlap assembly [15,16].

Definition 4.8. Let $x, y \in \Sigma^*$. The *2-overlap catenation* of x and y, $x\overline{\odot}^2 y$, is defined as

$$x\overline{\odot}^2 y = \{z \in \Sigma^+ \mid (\exists u, w \in \Sigma^*)(\exists v \in \Sigma^{\geq 2})\ x = uv, y = vw, z = uvw\}.$$

2-overlap catenation is extended in the natural to an operation on languages. For $L \subseteq \Sigma^*$, we define inductively $2\mathbb{OC}^{(0)}(L) = L$ and $2\mathbb{OC}^{(i+1)}(L) = 2\mathbb{OC}^{(i)}(L)\overline{\odot}^2 2\mathbb{OC}^{(i)}(L)$, $i \geq 0$. The *2-overlap catenation closure* of L is $2\mathbb{OC}^*(L) = \bigcup_{i=0}^{\infty} 2\mathbb{OC}^{(i)}(L)$.

Due to commutativity of unary languages we get the following property which will be crucial for establishing closure of unary regular languages under outfix-guided insertion closure.

Lemma 4.9. *If $x, y \in a^*$ are unary strings, then $x \leftarrow y = x\overline{\odot}^2 y$.*

Proof. Consider $w \in (x \leftarrow y)$, that is, we can write $w = x_1uzvx_2$, where $x = x_1uvx_2$, $y = uzv$ and $u, v \neq \varepsilon$. Since concatenation of unary strings is commutative, we have

$$w = x_1x_2uvz, \text{ where } x = x_1x_2uv, \quad y = uvz.$$

This establishes that $w \in x\overline{\odot}^2 y$.

Conversely, consider $z \in x\overline{\odot}^2 y$, that is, $z = uvw$ where $x = uv$, $y = vw$ and $|v| \geq 2$. Write $v = v_1 v_2$ where $v_1, v_2 \in \Sigma^+$. Now, again relying just on commutativity of unary concatenation, $z = uv_1 wv_2 \in x \leftarrow y$. □

Corollary 4.10. *If L is a unary language then $\mathbb{OGI}^*(L) = 2\mathbb{OC}^*(L)$.*

The 2-overlap closure of a regular language is always regular. The construction does not depend on a language being unary, so we state the result for regular languages over an arbitrary alphabet. Csuhaj-Varju et al. [15] have shown that iterated overlap assembly preserves regularity. The proof of Lemma 4.11 is inspired by Theorem 4 of [15] but does not follow from it because [15] defines iteration of operations as right one-sided iteration and, furthermore, 2-overlap catenation has an additional length restriction on the overlapping strings.

Lemma 4.11. *The 2-overlap catenation closure of a regular language is regular.*

Proof. Consider a regular language L recognized by an NFA $A = (\Sigma, Q, \delta, q_0, F)$. We construct for the language $2\mathbb{OC}^*(L)$ an NFA $B = (\Sigma, 2^Q, \gamma, \{q_0\}, 2^F - \{\emptyset\})$ where the transitions of γ are defined below.

For $\emptyset \neq P \subseteq Q$ and $b \in \Sigma$ define

$$\gamma(P, b) = \{(\delta(P, b) - X_{\text{rem}}) \cup Y_{\text{add}} \mid X_{\text{rem}} \subseteq F,\ \delta(P, b) - X_{\text{rem}} \neq \emptyset,\ Y_{\text{add}} \subseteq \delta(q_0, b)\}.$$

States of B are subsets of Q and the transition relation is nondeterministic: $\gamma(P, b)$ is a collection of subsets of Q.

The computation of B simulates multiple computations of A. When reading a symbol $b \in \Sigma$, the NFA B can guess that this occurrence of b begins (one or more) 2-overlap-catenated strings, and adds to the simulated computations the corresponding states of $\delta(q_0, b)$. Always when a simulated computation reaches a state of F, the NFA B can nondeterministically guess that b ends a string that is 2-overlap concatenated with another string. This is done by the choice of the set $X_{\text{rem}} \subseteq F$ in the definition of γ. Note that the condition $\delta(P, b) - X_{\text{rem}} \neq \emptyset$ guarantees that at least one of the simulated computations that were originated before reading b must remain alive: this enforces that the overlap with the new computations indeed will be at least two.

It is clear that, by always choosing the sets X_{rem} and Y_{add} correctly, B has a computation on an arbitrary string in $w \in 2\mathbb{OC}^*(L(A))$ that ends in a state $P \subseteq F$, $P \neq \emptyset$. The set P consists of final states of A that appear in an accepting computation in all strings that in the representation of w as a 2-overlap catenation of strings of $L(A)$ are a suffix of w. Note that the construction works also if w has length one: in this case w must be an element of $L(A)$.

To verify the converse inclusion $L(B) \subseteq 2\mathbb{OC}^*(L(A))$, we note that, in general, some parts of computations of B need not simulate any iterated 2-overlap concatenation of a set of strings of $L(A)$. For example, if A accepts both xyz and y where $|y| \geq 2$, on the string xyz the NFA B can begin a second computation C_2 when reading the first symbol of y and this computation then may end in a final state at the end of the substring y. However, the existence of the superfluous computation C_2 cannot lead to new illegal computations because the transitions of γ add new computations depending only on the input symbol and not on the current state. (In the transitions of γ, the sets Y_{add} depend only on the input symbol and the initial state of A.) Thus, the added superfluous computations cannot cause B to accept strings not in $2\mathbb{OC}^*(L(A))$. □

By Corollary 4.10 and Lemma 4.11 we have shown that unary regular languages are closed under outfix-guided insertion closure, contrasting the result of Theorem 4.6 for general regular languages.

Theorem 4.12. *The outfix-guided insertion closure of a unary regular language is always regular.*

4.1. One-sided iterated outfix-guided insertion

The left and right one-sided insertion closures are restricted variants of the general outfix-guided insertion closure, so Theorem 4.6 does not directly imply the existence of regular languages L_1 and L_2 such that $\mathbb{LOGI}^*(L_1, L_2)$ or $\mathbb{ROGI}^*(L_1, L_2)$ are non-regular. Here we show that the one-sided outfix-guided insertion closures are not, in general, regularity preserving. For the left-one one-sided outfix-guided insertion closure the construction is similar to that used in the proof of Theorem 4.6. However, this construction does not work for right one-sided closure because if L is the language used in the proof of Theorem 4.6, then $\mathbb{ROGI}^*(L, L)$ is the finite language $L \cup \{\$a_3a_1a_2b_1b_3\$\}$.

Proposition 4.13. *There exist finite languages L_1, L_2, L_3 and L_4 such that $\mathbb{ROGI}^*(L_1, L_2)$ and $\mathbb{LOGI}^*(L_3, L_4)$ are non-regular.*

Proof. We consider first the right one-sided outfix-guided insertion closure. Let $\Sigma = \{\$, a, b, c, d\}$ and choose $L_2 = \{a\$b\}$, $L_1 = \{acdb, cabd\}$. Then

$$\mathbb{ROGI}^*(L_1, L_2) = \{(ca)^i\$(bd)^i \mid i \geq 0\} \cup \{a(ca)^i\$(bd)^ib \mid i \geq 0\},$$

which is non-regular. Inserting $a\$b$ into $cabd$ derives $ca\$bd$ and next, in a right one-sided derivation, inserting the latter string into $acdb$ derives $aca\$bdb$. Continuing in this way we get a right one-sided derivation for all strings in the set appearing on the right side of the equation.

The fact that $\mathbb{ROGI}^*(L_1, L_2)$ does not contain any additional strings follows from the property of right one-sided iterated insertions: all strings that are inserted into L_1 will have the marker \$ and strings of L_1 cannot be inserted into strings of L_1. We leave to the reader the details of verifying that the inclusion holds from left to right.

The construction of the languages L_3 and L_4 for the left one-sided outfix-guided insertion closure is obtained by modifying the language in the proof of Theorem 4.6. Let $\Sigma = \{a_1, a_2, a_3, b_1, b_2, b_3\}$ and define $L_3 = \{\$a_3a_1b_1b_3\$\}$ and

$$L_4 = \{a_3a_1a_2b_1,\ a_2b_2b_1b_3,\ a_1a_2a_3b_2,\ a_3b_3b_2b_1,\ a_2a_3a_1b_3,\ a_1b_1b_3b_2\}.$$

Denote

$$L_5 = \{\$a_3(a_1a_2a_3)^i z(b_3b_2b_1)^i b_3\$ \mid i \geq 0,\ z \in S_{\text{middle}}\},$$

where $S_{\text{middle}} = \{a_1b_1,\ a_1a_2b_1,\ a_1a_2b_2b_1,\ a_1a_2a_3b_2b_1,\ a_1a_2a_3b_3b_2b_1,\ a_1a_2a_3a_1b_3b_2b_1\}$.

From the proof of Theorem 4.6 it follows that

$$\mathbb{LOGI}^*(L_3, L_4) = L_5.$$

Note that the first part of the proof of Theorem 4.6 establishes that all strings of L_5 are obtained by left one-sided iterated insertion of L_4 into $\$a_3a_1b_1b_3\$$. Thus, $L_5 \subseteq \mathbb{LOGI}^*(L_3, L_4)$. The proof of Theorem 4.6 also establishes that $\mathbb{OGI}^*(L_3 \cup L_4) = L_5$ and directly by the definition of the iterated operations, $\mathbb{LOGI}^*(L_3, L_4) \subseteq \mathbb{OGI}^*(L_3 \cup L_4)$. □

5. Outfix-guided insertion and context-free languages

It is well known that the family of context-free languages is closed under ordinary insertion. We show that context-free languages are not closed under outfix-guided (or prefix-guided, suffix-guided, respectively) insertion. This contrasts also the corresponding result for regular languages from Lemma 4.1.

Theorem 5.1. *There exists a context-free language L such that $L \leftarrow L$ is not context-free.*

Proof. Let $\Sigma = \{\$, a, b, c\}$. By choosing

$$L = \{\$a^n\$\$c^n \mid n \geq 1\} \cup \{\$a^n\$b^n\$ \mid n \geq 1\}$$

we note that

$$(L \leftarrow L) \cap \$a^+\$b^+\$c^+ = \{\$a^n\$b^n\$c^n \mid n \geq 1\}.$$

The claim follows since the intersection of a context-free language and a regular language is always context-free [22]. □

The same language L as in the proof of Theorem 5.1 can be used to establish that context-free languages are not closed under prefix-guided insertion and the reversal of L can be used to establish non-closure under suffix-guided insertion.

Corollary 5.2. *There exist context-free languages L_1 and L_2 such that $L_1 \overset{\text{pgi}}{\leftarrow} L_1$ and $L_2 \overset{\text{sgi}}{\leftarrow} L_2$ are not context-free.*

On the other hand, the outfix-guided insertion of a regular (respectively, context-free) language into a context-free (respectively, regular) language is always context-free.

Theorem 5.3. *If L_1 is context-free and L_2 is regular, then $L_1 \leftarrow L_2$ and $L_2 \leftarrow L_1$ are context-free.*

Proof. Suppose L_1 is recognized by a nondeterministic PDA M and L_2 is recognized by an NFA A. By combining the finite state transitions of M and A as in the proof of Lemma 4.1, and simultaneously simulating the pushdown stack of M we can construct a PDA M_1 for $L_1 \leftarrow L_2$. Always when M_1 makes a transition simulating a transition of M, it makes a corresponding stack operation. On the other hand, transitions of M_1 simulating only transitions of A do not touch the stack. A PDA for $L_2 \leftarrow L_1$ is obtained by interchanging in the construction of Lemma 4.1 the roles of M and A. □

The analogy of Theorem 5.3 does not hold for deterministic context-free languages. Techniques for proving that a language is not deterministic context-free are known already from [25].

Theorem 5.4. *If L_1 is deterministic context-free and L_2 is regular, the languages $L_1 \leftarrow L_2$ or $L_2 \leftarrow L_1$ need not be deterministic context-free.*

Proof. First we show that there exist a DCFL L_1 and a regular language L_2 such that $L_1 \leftarrow L_2$ is not deterministic context-free.

Let $L_1 = \{cda^ib^ia^j \mid i, j \geq 1\} \cup \{ca^ib^ja^j \mid i, j \geq 1\}$ and $L_2 = \{cda\}$. We can outfix-guided insert cda in a non-trivial way only into words of the form $ca^ib^ja^j$, which gives us

$$L_1 \leftarrow L_2 = cd \cdot (\{a^ib^ia^j \mid i, j \geq 1\} \cup \{a^ib^ja^j \mid i, j \geq 1\}).$$

From [25], we have that $L \subseteq (\Sigma - c)^*$ is a DCFL if and only if cL is a DCFL and that the language $(\{a^ib^ia^j \mid i, j \geq 1\} \cup \{a^ib^ja^j \mid i, j \geq 1\})$ is not deterministic. Thus, $L_1 \leftarrow L_2$ is not a DCFL.

Second, we show that there exist a regular language L_3 and a DCFL L_4 such that $L_3 \leftarrow L_4$ is not deterministic context-free.

Let $L_3 = (a^*bac) + (aba^*)$ and $L_4 = \{b^ja^jc \mid j \geq 1\} \cup \{a^ib^ia^2 \mid i \geq 1\}$. We can insert words of the form b^ja^jc in a non-trivial way only into words of the form a^ibac. This gives us the set $\{a^ib^ja^jc \mid i, j \geq 1\}$. Similarly, we can only insert words of the form $a^ib^ia^2$ into words of the form aba^2a^j, resulting in the set $\{a^ib^ia^j \mid i \geq 1, j \geq 2\}$. Thus,

$$L_3 \leftarrow L_4 = \{a^ib^ja^jc \mid i, j \geq 1\} \cup \{a^ib^ia^j \mid i \geq 1, j \geq 2\},$$

which is not deterministic context-free. □

Theorem 5.1 raises the question how complex languages can be obtained from context-free languages using iterated outfix-guided insertion. Note that if L_1 and L_2 are context-free, it is easy to verify that $L_1 \leftarrow L_2$ is deterministic context-sensitive. Next we consider the corresponding question for the insertion closures.

Proposition 5.5. *If L_1 and L_2 are context-free then $\mathbb{ROGI}^*(L_1, L_2)$ and $\mathbb{LOGI}^*(L_1, L_2)$ are context-sensitive.*

Proof. We consider only the right one-sided insertion closure – the proof for left one-sided insertion closure is similar.

Below by a substring occurrence of w we mean a unique substring beginning at a specified position in w. From the definition of right one-sided iterated insertion it follows that $w \in \mathbb{ROGI}^*(L_1, L_2)$ if and only if there exists $k \geq 1$ and a sequence of substring occurrences of w: $s_1, s_2, \ldots, s_k$ where $s_k = w$ and s_i is always inside the substring occurrence s_{i+1}, $i = 1, \ldots, k-1$, and:

- We can write $s_1 = x_1uzvx_2$, where $x_1uvx_2 \in L_1$, $uzv \in L_2$.
- We can write $s_2 = x_1'u'z'v'x_2'$, where $x_1'u'v'x_2' \in L_1$, $u'z'v' = s_1$,
- ...
- We can write $s_k = x_1''u''z''v''x_2''$, where $x_1''u''v''x_2'' \in L_1$, $u''z''v'' = s_{k-1}$.

Furthermore, we can assume that $|s_i| < |s_{i+1}|$, $i = 1, \ldots, k-1$, because if this is not the case, in the chain we can simply omit s_{i+1}. Now, on input w, a nondeterministic linear space Turing machine M can begin by guessing s_1 and verifying that it has the required decomposition. In the $(i+1)$st stage M always "remembers" (by markers on the tape) the previous string s_i, then guesses the substring s_{i+1} (where s_i is a substring of s_{i+1}) and verifies that the conditions hold for s_{i+1}. At the end M accepts if $s_k = w$. Since the values $|s_i|$ form a strictly increasing sequence, the process can be ended after at most $|w|$ stages. □

In the proof of Proposition 5.5 it is sufficient to know that the languages L_1 and L_2 are context-sensitive, and as a consequence it follows that context-sensitive languages are closed under one-sided outfix-guided insertion closure.

We conjecture that, for any context-free language L, $\mathbb{OGI}^*(L)$ must be context-sensitive. Constructing a linear bounded automaton for $\mathbb{OGI}^*(L)$ is more difficult than in the case of the right or left one-sided insertion closures, because a direct simulation of a derivation of $w \in \mathbb{OGI}^*(L)$ (i.e., simulation of the iterated outfix-guided insertion steps producing w) would need to remember, at a given time, an unbounded number of substrings of the input.

Also we do not know how to make the procedure in the proof of Proposition 5.5 deterministic and it remains open whether the one-sided outfix-guided insertion closures of context-free languages are always deterministic context-sensitive.

6. Deciding closure under outfix-guided insertion

In this section we consider the question whether a given language is closed under outfix-guided insertion and show that this question is undecidable for context-free languages.

We say that a language L is *closed* under outfix-guided insertion, or *og-closed* for short, if outfix-guided inserting strings of L into L does not produce strings outside of L, that is, $(L \leftarrow L) \subseteq L$.

A natural algorithmic problem is then to decide for a given language L whether or not L is og-closed. If L is regular, by Lemma 4.1, we can decide whether or not L is og-closed. For a given DFA A, Lemma 4.1 yields only an NFA for the language $L(A) \leftarrow L(A)$. In general, the NFA equivalence or inclusion problem is PSPACE complete [23]. However, inclusion of an NFA language in the language $L(A)$ can be tested efficiently when A is deterministic.

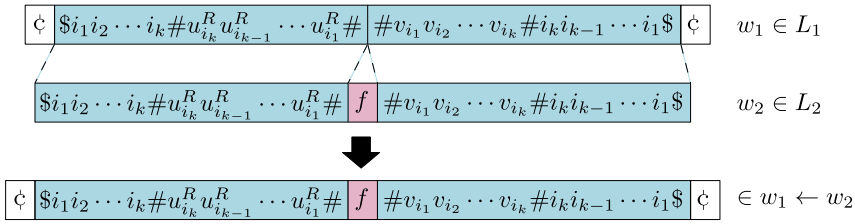


Fig. 4. Only possible outfix-guided insertion of w_2 into w_1.

Theorem 6.1. *There is a polynomial time algorithm to decide whether for a given DFA A the language $L(A)$ is og-closed.*

Proof. As in the proof of Lemma 4.1 we construct an NFA B for the language $L(A) \leftarrow L(A)$. The number of states of B is quadratic in the number of states of A. Let A' be the DFA obtained from A by interchanging the final states and the non-final states. Now $L(B) \subseteq L(A)$ if and only if $L(B) \cap L(A') = \emptyset$ and intersection emptiness for NFAs can be tested in polynomial time. □

The method used in Theorem 6.1 does not yield an efficient algorithm if the regular language L is specified by an NFA. The complexity of deciding og-closure of a language accepted by an NFA remains open. On the other hand, using a reduction from the Post Correspondence Problem it follows that the question whether or not a context-free language is og-closed is undecidable.

Theorem 6.2. *For a context-free language L, specified e.g. by a context-free grammar, the question whether or not L is og-closed is undecidable.*

Proof. Recall that an instance of the *Post Correspondence Problem* (PCP) [22] consists of two lists of strings $((u_1, \ldots, u_n), (v_1, \ldots, v_n))$, $u_i, v_i \in \Sigma^*$, $1 \le i \le n$, and a solution of this instance is a sequence of integers $(i_1, \ldots, i_k)$, $i_j \in \{1, \ldots, n\}$, $j = 1, \ldots, k$, such that $u_{i_1} \cdots u_{i_k} = v_{i_1} \cdots v_{i_k}$. It is well known that deciding whether or not a PCP instance has a solution is undecidable [22].

Let $I_{\text{PCP}} = ((u_1, \ldots, u_n), (v_1, \ldots, v_n))$, $u_i, v_i \in \{a, b\}^*$, $1 \le i \le n$ be an arbitrary instance of PCP. Choose $\Sigma = \{a, b, f, \dot{\varsigma}, \$, \#\}$ and define

$$L_1 = \{\dot{\varsigma}\$i_1 i_2 \cdots i_r \# u_{i_r}^R u_{i_{r-1}}^R \cdots u_{i_1}^R \#\# v_{j_1} v_{j_2} \cdots v_{j_s} \# j_s j_{s-1} \cdots j_1 \$\dot{\varsigma} \mid r, s \ge 1, 1 \le i_x, j_y \le n, 1 \le x \le r, 1 \le y \le s\}, \text{ and,}$$

$$L_2 = \{\$i_1 i_2 \cdots i_r \# w \# f \# w^R \# i_r i_{r-1} \cdots i_1 \$ \mid w \in \{a, b\}^*, r \ge 1, 1 \le i_x \le n, 1 \le x \le r\}.$$

The languages L_1 and L_2 are context-free. (The language L_2 can be generated by a linear context-free grammar and L_1 is the concatenation of two linear context-free languages.)

We define $L = L_1 \cup L_2$ and claim that the instance I_{PCP} has a solution if and only if L is not og-closed. Below we prove both implications of the claim.

- "I_{PCP} has a solution implies L is not closed": Suppose that $(i_1, \ldots, i_k)$ is a solution for I_{PCP}. Now

  $$w_1 = \dot{\varsigma}\$i_1 i_2 \cdots i_k \# u_{i_k}^R u_{i_{k-1}}^R \cdots u_{i_1}^R \#\# v_{i_1} v_{i_2} \cdots v_{i_k} \# i_k i_{k-1} \cdots i_1 \$\dot{\varsigma} \in L_1 \ (\subseteq L).$$

  Also since $(i_1, \ldots, i_k)$ is solution we note that

  $$w_2 = \$i_1 i_2 \cdots i_k \# u_{i_k}^R u_{i_{k-1}}^R \cdots u_{i_1}^R \# f \# v_{i_1} v_{i_2} \cdots v_{i_k} \# i_k i_{k-1} \cdots i_1 \$ \in L_2 \ (\subseteq L).$$

  As illustrated in Fig. 4, the string w_2 can be (in a unique way) outfix-guided inserted into the string w_1 and the resulting string is not in L because no string of L contains both symbols $\dot{\varsigma}$ and f.
- "I_{PCP} has no solution implies L is closed": Recall that by a trivial outfix-guided derivation step we mean a derivation step $w \overset{[w]}{\Rightarrow} w$ where w is obtained from itself by selecting the outfix to consist of a prefix and suffix of w whose concatenation is equal to w.
  Using the assumption that the instance I_{PCP} does not have a solution we show that strings of L can be inserted into strings of L using only trivial derivation steps which naturally then implies $(L \leftarrow L) \subseteq L$.
  Strings of L_1 begin and end with the symbol $\dot{\varsigma}$ and this symbol occurs exactly two times in strings of L_1. Thus, strings of L_1 can be outfix-guided inserted into strings of L_1 only using a trivial derivation step. For the same reason (by replacing $\dot{\varsigma}$ with \$) strings of L_2 can be inserted into strings of L_2 only using a trivial derivation step.

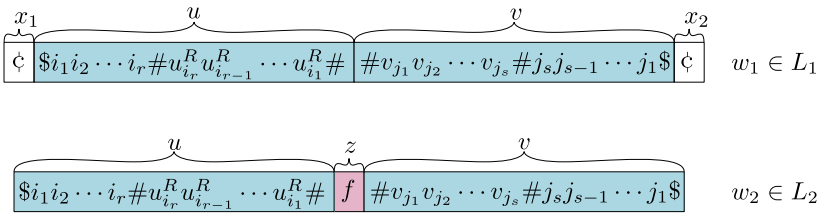


Fig. 5. Decompositions of $w_1 \in L_1$ and $w_2 \in L_2$.

Strings of L_1 cannot be outfix-guided inserted into strings of L_2 because the former begin and end with the symbol ċ and the latter do not contain any occurrences of ċ. The remaining possibility we need to consider is under what conditions strings of L_2 can be outfix-guided inserted into strings of L_1.
Consider

$$w_1 = \dot{c}\$i_1 i_2 \cdots i_r \# u_{i_r}^R u_{i_{r-1}}^R \cdots u_{i_1}^R \# \# v_{j_1} v_{j_2} \cdots v_{j_s} \# j_s j_{s-1} \cdots j_1 \$\dot{c} \in L_1,$$

$$\text{and, } w_2 = \$i_1 i_2 \cdots i_r \# w \# f \# w^R \# i_r i_{r-1} \cdots i_1 \$ \in L_2,$$

and suppose we can write $w_1 = x_1 u v x_2$, $w_2 = uzv$, $u, v \neq \varepsilon$. Since w_1 does not contain occurrences of the symbol f, in the decomposition of w_2 the symbol f must be in the substring z. The string w_2 begins and ends with \$ which are then the first and last symbol of u and v, respectively (and consequently $x_1 = x_2 = \dot{c}$). Since the concatenation of u and v must contain all four occurrences of # in w_1, it follows that

$$u = \$i_1 i_2 \cdots i_r \# u_{i_r}^R u_{i_{r-1}}^R \cdots u_{i_1}^R \# \text{ and } v = \# v_{j_1} v_{j_2} \cdots v_{j_s} \# j_s j_{s-1} \cdots j_1 \$.$$

Now in the decomposition of w_2 the only possibility is that $z = f$, and the strings w_1 and w_2 must be as illustrated in Fig. 5. From the definition of the language L_2 it follows that $r = s$, $i_x = j_x$, $x = 1, \ldots, r$ and $(u_{i_r}^R u_{i_{r-1}}^R u_{i_1}^R)^R = v_{i_1} v_{i_2} \cdots v_{i_r}$. Together these conditions mean that $(i_1, \ldots, i_r)$ is a solution for the instance I_{PCP}, which contradicts our assumption that the instance did not have a solution. □

Note that in the proof of Theorem 6.2 the language L_1 is not deterministic context-free. It remains open whether og-closure can be decided for deterministic context-free languages.

7. Conclusion

Analogously with the recent overlap assembly operation [15,16], we have introduced an overlapping insertion operation on strings and have studied closure and decision properties of the outfix-guided insertion operation. While closure properties of non-iterated outfix-guided insertion are straightforward to establish, the questions become more involved for the outfix-guided insertion closure. As the main result we have shown that the outfix-guided insertion closure of a finite language need not be regular.

Much work remains to be done on outfix-guided insertion. One of the main open questions is to determine upper bounds for the complexity of the outfix-guided insertion closures of regular languages. Does there exist regular languages L such that the outfix-guided insertion closure of L is non-context-free?

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